Risk Aversion and the Labor Margin in Dynamic Equilibrium Models

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Coefficient of Relative Risk Aversion

Suppose a household has preferences:

\[ E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t), \]

\[ u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta l_t \]

What is the household’s coefficient of relative risk aversion?
Suppose a household has preferences:

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\[ u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta l_t \]

What is the household’s coefficient of relative risk aversion?

Answer: 0
Coefficient of Relative Risk Aversion

Suppose a household has preferences:

\[
E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t),
\]

\[
u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1 - \gamma} - \eta \frac{l_t^{1+\chi}}{1 + \chi}
\]

What is the household’s coefficient of relative risk aversion?

Answer: \[
\frac{1}{1 + \frac{1}{\gamma + \chi}}
\]
Outline of Presentation

- Define risk aversion rigorously in dynamic equilibrium models
- Derive closed-form expressions
- Show the labor margin can have big effects on risk aversion
- Compute numerical solutions far away from steady state
- Relate risk aversion to asset pricing, stochastic discount factor
Outline of Presentation

- Define risk aversion rigorously in dynamic equilibrium models
- Derive closed-form expressions
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See the paper for:

- Epstein-Zin preferences
- Internal, external habits
A Household

Household preferences:

\[ E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau, l_\tau), \]

Flow budget constraint:

\[ a_{\tau+1} = (1 + r_\tau)a_\tau + w_\tau l_\tau + d_\tau - c_\tau, \]

No-Ponzi condition:

\[ \lim_{T \to \infty} \prod_{\tau=t}^{T} (1 + r_{\tau+1})^{-1} a_{\tau+1} \geq 0, \]

\{w_\tau, r_\tau, d_\tau\} are exogenous processes, governed by \( \theta_\tau \)
State variables of the household’s problem are \((a_t; \theta_t)\).

Let:

\[
\begin{align*}
    c_t^* & \equiv c^*(a_t; \theta_t), \\
    l_t^* & \equiv l^*(a_t; \theta_t).
\end{align*}
\]
The Value Function

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\end{align*}
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Value function, Bellman equation:

\[
V(a_t; \theta_t) = u(c_t^*, l_t^*) + \beta E_t V(a_{t+1}^*; \theta_{t+1}),
\]

where:

\[
a_{t+1}^* \equiv (1 + r_t)a_t + w_t l_t^* + d_t - c_t^*.
\]
Assumption 1. *The function* $u(c_t, l_t)$ *is increasing in its first argument, decreasing in its second, twice-differentiable, and strictly concave.*

Assumption 2. *The value function* $V : X \rightarrow \mathbb{R}$ *for the household’s optimization problem exists and satisfies the Bellman equation*

$$V(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} u(c_t, l_t) + \beta E_t V(a_{t+1}; \theta_{t+1}).$$

Assumption 3. *For any* $(a_t; \theta_t) \in X$, *the household’s optimal choice* $(c^*_t, l^*_t)$ *lies in the interior of* $\Gamma(a_t; \theta_t)$.

Assumption 4. *The value function* $V(\cdot; \cdot)$ *is twice-differentiable in its first argument. (It then follows that* $c^*$, $l^*$ *are differentiable.*)
Assumptions about the Economic Environment

**Assumption 5.** *The household is atomistic.*

**Assumption 6.** *The household is representative.*

**Assumption 7.** *The model has a nonstochastic steady state, \( x_t = x_{t+k} \) for \( k = 1, 2, \ldots \), and \( x \in \{ c, l, a, w, r, d, \theta \} \).*
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Assumption 7'. *The model has a balanced growth path that can be renormalized to a nonstochastic steady state after a suitable change of variables.*
Arrow-Pratt in a Static One-Good Model (Review)

Compare:

\[ E u(c + \sigma \varepsilon) \quad \text{vs.} \quad u(c - \mu) \]
Arrow-Pratt in a Static One-Good Model (Review)

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\[ E u(c + \sigma \epsilon) \quad \text{vs.} \quad u(c - \mu) \]

Compute:

\[ u(c - \mu) \approx u(c) - \mu u'(c), \]
Compare:

\[ E u(c + \sigma \varepsilon) \text{ vs. } u(c - \mu) \]

Compute:

\[ u(c - \mu) \approx u(c) - \mu u'(c), \]
\[ E u(c + \sigma \varepsilon) \approx u(c) + u'(c)\sigma E[\varepsilon] + \frac{1}{2} u''(c)\sigma^2 E[\varepsilon^2], \]
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\[ \mu = \frac{-u''(c)}{u'(c)} \frac{\sigma^2}{2}. \]
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\[ \mu = \frac{-u''(c)}{u'(c)} \frac{\sigma^2}{2}. \]

Coefficient of absolute risk aversion is defined to be:

\[ \lim_{\sigma \to 0} 2\mu(\sigma)/\sigma^2 = \frac{-u''(c)}{u'(c)}. \]
Arrow-Pratt in a Dynamic Model

Consider a one-shot gamble in period $t$:

$$a_{t+1} = (1+r_t)a_t + w_t + d_t - c_t + \sigma \epsilon_{t+1}, \quad (\ast)$$

Note we cannot easily consider gambles over:

- $a_t$ (state variable, already known at $t$)
- $c_t$ (choice variable)

Note $(\ast)$ is equivalent to gamble over income:

$$a_{t+1} = (1+r_t)a_t + w_t + (d_t + \sigma \epsilon_t + 1) - c_t,$$

or asset returns:

$$a_{t+1} = (1+r_t+\sigma \tilde{\epsilon}_t)a_t + w_t + d_t - c_t.$$
Arrow-Pratt in a Dynamic Model

Consider a one-shot gamble in period $t$:

$$a_{t+1} = (1 + r_t) a_t + w_t l_t + d_t - c_t + \sigma \varepsilon_{t+1}, \quad (\star)$$
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Welfare loss from $\mu$:

$$\beta E_t V_1(a^*_{t+1}; \theta_{t+1}) \mu.$$
Arrow-Pratt in a Dynamic Model

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$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t - \mu.$$

Welfare loss from $\mu$:

$$\beta E_t V_1(a^*_t; \theta_{t+1}) \mu.$$

Loss from $\sigma$:

$$\beta E_t V_{11}(a^*_t; \theta_{t+1}) \frac{\sigma^2}{2}.$$
Definition 1. The household’s coefficient of absolute risk aversion at $(a_t; \theta_t)$ is given by $R^a(a_t; \theta_t) = \lim_{\sigma \to 0} \frac{2\mu(\sigma)}{\sigma^2}$. 
Definition 1. The household’s coefficient of absolute risk aversion at \((a_t; \theta_t)\) is given by \(R^a(a_t; \theta_t) = \lim_{\sigma \to 0} 2\mu(\sigma)/\sigma^2\).

Proposition 1. The household’s coefficient of absolute risk aversion at \((a_t; \theta_t)\) satisfies

\[
R^a(a_t; \theta_t) = \frac{-E_tV_{11}(a_{t+1}^*; \theta_{t+1})}{E_tV_1(a_{t+1}^*; \theta_{t+1})}.
\]
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Proposition 1. The household’s coefficient of absolute risk aversion at \((a_t; \theta_t)\) satisfies

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R^a(a_t; \theta_t) = \frac{-E_t V_{11}(a^*_t+1; \theta_{t+1})}{E_t V_1(a^*_t+1; \theta_{t+1})}.
\]

Coefficient of Absolute Risk Aversion

**Definition 1.** The household’s coefficient of absolute risk aversion at \((a_t; \theta_t)\) is given by \(R^a(a_t; \theta_t) = \lim_{\sigma \to 0} \frac{2\mu(\sigma)}{\sigma^2}\).

**Proposition 1.** The household’s coefficient of absolute risk aversion at \((a_t; \theta_t)\) satisfies

\[
R^a(a_t; \theta_t) = -\frac{E_t V_{11}(a^*_{t+1}; \theta_{t+1})}{E_t V_1(a^*_{t+1}; \theta_{t+1})}.
\]

Evaluated at the nonstochastic steady state, this simplifies to:

\[
R^a(a; \theta) = -\frac{V_{11}(a; \theta)}{V_1(a; \theta)}.
\]

Solve for $V_1$ and $V_{11}$

Benveniste-Scheinkman:

$$V_1(a_t; \theta_t) = (1 + r_t) u_1(c_t^*, l_t^*).$$

(*)
Solve for $V_1$ and $V_{11}$

Benveniste-Scheinkman:

$$V_1(a_t; \theta_t) = (1 + r_t) u_1(c_t^*, l_t^*). \quad (*)$$

Differentiate $(*)$ to get:

$$V_{11}(a_t; \theta_t) = (1 + r_t) \left[ u_{11}(c_t^*, l_t^*) \frac{\partial c_t^*}{\partial a_t} + u_{12}(c_t^*, l_t^*) \frac{\partial l_t^*}{\partial a_t} \right].$$
Solve for $\partial l_t^* / \partial a_t$ and $\partial c_t^* / \partial a_t$
Solve for $\partial l_t^*/\partial a_t$ and $\partial c_t^*/\partial a_t$

Household intratemporal optimality: $-u_2(c_t^*, l_t^*) = w_t u_1(c_t^*, l_t^*)$. 
Solve for $\frac{\partial l^*_t}{\partial a_t}$ and $\frac{\partial c^*_t}{\partial a_t}$

Household intratemporal optimality: $-u_2(c^*_t, l^*_t) = w_t \ u_1(c^*_t, l^*_t)$.

Differentiate to get:

$$\frac{\partial l^*_t}{\partial a_t} = -\lambda_t \frac{\partial c^*_t}{\partial a_t},$$

$$\lambda_t \equiv \frac{w_t u_{11}(c^*_t, l^*_t) + u_{12}(c^*_t, l^*_t)}{u_{22}(c^*_t, l^*_t) + w_t u_{12}(c^*_t, l^*_t)}.$$
Solve for $\frac{\partial l^*_t}{\partial a_t}$ and $\frac{\partial c^*_t}{\partial a_t}$

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Household Euler equation:

$$u_1(c^*_t, l^*_t) = \beta E_t(1 + r_{t+1}) u_1(c^*_{t+1}, l^*_{t+1}),$$
Solve for $\frac{\partial l^*_t}{\partial a_t}$ and $\frac{\partial c^*_t}{\partial a_t}$

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\[-u_2(c^*_t, l^*_t) = w_t \, u_1(c^*_t, l^*_t).\]

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\[
u_1(c^*_t, l^*_t) = \beta E_t(1 + r_{t+1}) \, u_1(c^*_{t+1}, l^*_{t+1}),
\]

Differentiate, substitute out for $\frac{\partial l^*_t}{\partial a_t}$, and use BC, TVC to get:
\[
\frac{\partial c^*_t}{\partial a_t} = \frac{r}{1 + w \lambda}.
\]
Solve for Coefficient of Absolute Risk Aversion

\[ V_1(a; \theta) = (1 + r) u_1(c, l), \]
Solve for Coefficient of Absolute Risk Aversion

\[ V_1(a; \theta) = (1 + r) u_1(c, l), \]
\[ V_{11}(a; \theta) = (1 + r) \left[ u_{11}(c, l) \frac{\partial c^*_t}{\partial a_t} + u_{12}(c, l) \frac{\partial l^*_t}{\partial a_t} \right], \]
Solve for Coefficient of Absolute Risk Aversion

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\[ \frac{\partial l^*_t}{\partial a_t} = -\lambda \frac{\partial c^*_t}{\partial a_t}, \]

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Solve for Coefficient of Absolute Risk Aversion

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\[ \frac{\partial c^*_t}{\partial a_t} = \frac{r}{1 + w \lambda}. \]

**Proposition 2.** The household’s coefficient of absolute risk aversion in Proposition 1, evaluated at steady state, satisfies:

\[ R^a(a; \theta) = \frac{-V_{11}(a; \theta)}{V_1(a; \theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{r}{1 + w \lambda}. \]
Corollary 3.

\[ R^a(a; \theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{r}{1 + w \lambda} \leq \frac{-u_{11}}{u_1} r. \]

If \( r < 1 \), then \( R^a(a; \theta) \) is also less than \(-u_{11}/u_1\).
Corollary 3.

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Corollary 4. The household’s coefficient of absolute risk aversion is 0 if and only if the discriminant \( u_{11}u_{22} - u_{12}^2 = 0 \).
Coefficient of Absolute Risk Aversion

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e.g.:

\[ u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta l_t. \]
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e.g.:

\[ u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta l_t. \]

\[ u(c_t, l_t) = c_t^{\theta} (\bar{l} - l_t)^{1-\theta}. \]
Relative Risk Aversion

Consider Arrow-Pratt gamble of general size $A_t$:

$$a_{t+1} = (1 + r_t) a_t + w_t l_t + d_t - c_t + A_t \sigma \varepsilon_{t+1},$$

vs.

$$a_{t+1} = (1 + r_t) a_t + w_t l_t + d_t - c_t - A_t \mu.$$
Relative Risk Aversion

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a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t + A_t \sigma \varepsilon_{t+1},
$$

vs.

$$
a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t - A_t \mu.
$$

Risk aversion coefficient for this gamble:

$$
- A_t E_t V_{11}(a_{t+1}^*; \theta_{t+1}) \over E_t V_1(a_{t+1}^*; \theta_{t+1}) \cdot \quad (*)
$$
Relative Risk Aversion

Consider Arrow-Pratt gamble of general size $A_t$:

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vs.

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t - A_t \mu.$$

Risk aversion coefficient for this gamble:

$$-A_t E_t V_{11}(a^*_{t+1}; \theta_{t+1})$$

$$E_t V_1(a^*_{t+1}; \theta_{t+1}).$$

\[ (*) \]

A natural benchmark for $A_t$ is household wealth at time $t$. 
Household Wealth

In DSGE framework, household wealth has more than one component:

- financial assets $a_t$
- present value of labor income, $w_t l_t$
- present value of net transfers, $d_t$
- present value of leisure, $w_t (\bar{l} - l_t)$?
Household Wealth

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Leisure, in particular, can be hard to define, e.g.,

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}$$

and $\bar{l}$ is arbitrary.
Household Wealth

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and $\bar{l}$ is arbitrary.

Different definitions of household wealth lead to different definitions of relative risk aversion.
Definition 2. The coefficient of relative risk aversion, $R^r(a_t; \theta_t)$, is given by ($\ast$), with $A_t \equiv (1 + r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t, \tau} (c^*_\tau + w_\tau (\bar{l} - l^*_\tau))$.

In steady state:

$$R^r(a; \theta) = \frac{-A V_{11}(a; \theta)}{V_1(a; \theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c + w(\bar{l} - l)}{1 + w\lambda}.$$
Two Coefficients of Relative Risk Aversion

Definition 2. The coefficient of relative risk aversion, $R^r(a_t; \theta_t)$, is given by (*), with $A_t \equiv (1 + r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau} (c^*_\tau + w_\tau (\bar{l} - l^*_\tau))$.

In steady state:

$$R^r(a; \theta) = \frac{-A V_{11}(a; \theta)}{V_1(a; \theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c + w (\bar{l} - l)}{1 + w \lambda}.$$  

Definition 3. The consumption-only coefficient of relative risk aversion, $R^c(a_t; \theta_t)$, is given by (*), with $	ilde{A}_t \equiv (1 + r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau} c^*_\tau$.

In steady state:

$$R^c(a; \theta) = \frac{-\tilde{A} V_{11}(a; \theta)}{V_1(a; \theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1 + w \lambda}.$$
Corollary 5.

i) $R_c^c(a; \theta)$ and the intertemporal elasticity of substitution are reciprocal if and only if $\lambda = 0$;

ii) $R_r^r(a; \theta)$ and the intertemporal elasticity of substitution are reciprocal if and only if $\lambda = (\bar{l} - l)/c$. 

Corollary 5.

i) \( R^c(a; \theta) \) and the intertemporal elasticity of substitution are reciprocal if and only if \( \lambda = 0 \);

ii) \( R^r(a; \theta) \) and the intertemporal elasticity of substitution are reciprocal if and only if \( \lambda = (\bar{l} - l)/c \).

Proof:

\[
IES = \frac{-u_1}{u_{11} - \lambda u_{12}} \frac{1}{c}
\]

\[
R^c(a; \theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1 + w\lambda}
\]

\[
R^r(a; \theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c + w(\bar{l} - l)}{1 + w\lambda}
\]
Example 1

Period utility function: \[ u(c_t, l_t) = \frac{(c_t^\chi (1 - l_t)^{1-\chi})^{1-\gamma}}{1 - \gamma} \]
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Relative risk aversion: \[ R^r(a; \theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c + w(1 - l)}{1 + w\lambda} \]
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\[ w = \frac{-u_2}{u_1} = \frac{1 - \chi}{\chi} \frac{c}{1 - l} \]
\[ u_{11} = \chi [(1-\gamma)\chi - 1] c^{(1-\gamma)\chi - 2} (1 - l)^{(1-\gamma)(1-\chi)} \]
\[ u_{12} = -\chi (1 - \chi)(1 - \gamma) c^{(1-\gamma)\chi - 1} (1 - l)^{(1-\gamma)(1-\chi)-1} \]
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Example 1

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\[ u(c_t, l_t) = \left( \frac{c_t^\chi (1 - l_t)^{1-\chi}}{1 - \gamma} \right)^{1-\gamma} \]

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\[ \lambda = \frac{wu_{11} + u_{12}}{u_{22} + wu_{12}} = \frac{1 - l}{c} \]

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Example 2

Period utility function: 

\[ u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi} \]
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Period utility function:  \[ u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi} \]

Relative risk aversion is:  \[ R^c(a; \theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1 + w\lambda} \]
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\[ \lambda = \frac{wu_{11}}{u_{22}} = \frac{\gamma l}{\chi c} \]

\[ R^c(a; \theta) = \frac{\gamma}{1 + \frac{\gamma wl}{\chi c}} \approx \frac{1}{\gamma + \frac{1}{\chi}}, \quad \text{using } c = wl + ra + d \approx wl. \]
Example 2

Coefficient of Relative Risk Aversion ($R_c$)

$\chi = 5$

$\chi = 4$

$\chi = 3$

$\chi = 2$

$\chi = 1$

$\chi = 0$

$\chi = \infty$

$\gamma$

0 1 2 3 4 5 6 7 8 9 10
Risk Aversion Away from the Steady State

\[ u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}, \quad \gamma = 2, \ \chi = 1.5 \]
Risk Aversion Away from the Steady State

\[ u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}, \quad \gamma = 2, \quad \chi = 1.5 \]

Standard RBC model:

\[ Y_t = A_t K_t^{1-\alpha} L_t^\alpha \]

\[ K_{t+1} = (1-\delta) K_t + Y_t - C_t \]

\[ C_t^{-\gamma} = \beta E_t (1+r_{t+1}) C_t^{-\gamma} \]

\[ \eta L_t^\chi / C_t^{-\gamma} = w_t \]

\[ r_t = (1-\alpha) Y_t / K_t - \delta \]

\[ w_t = \alpha Y_t / L_t \]

\[ \log A_t = \rho \log A_{t-1} + \varepsilon_t \]
Risk Aversion Away from the Steady State

Auxiliary equations:

\[ R^a(a_t; \theta_t) = \frac{-E_t V_{11}(a_{t+1}^*; \theta_{t+1})}{E_t V_1(a_{t+1}^*; \theta_{t+1})} \]
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\[ \frac{\partial c^*_t}{\partial a_t} = \beta E_t (1 + r_{t+1}) \frac{\partial c^*_{t+1}}{\partial a_t} \]
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\[ \frac{\partial c^*_t}{\partial a_t} = \beta E_t (1 + r_{t+1}) \frac{\partial c^*_{t+1}}{\partial a^*_{t+1}} \left[ (1 + r_t) + w_t \frac{\partial l^*_t}{\partial a_t} - \frac{\partial c^*_t}{\partial a_t} \right] \]
Risk Aversion Away from the Steady State

Numerical solution to the model:
Risk Aversion Away from the Steady State

Numerical solution to the model:

Dashed black line denotes closed-form value of .9143.
(Compare to $\gamma = 2$)
Same exercise with Epstein-Zin Preferences (higher risk aversion):
Absolute risk aversion is countercyclical:
Asset Pricing

Price of an asset at time $t$:

$$E_t m_{t+1} p_{t+1}$$
Asset Pricing

Price of an asset at time $t$:

$$E_t m_{t+1} p_{t+1}$$

Lucas-Breeden stochastic discount factor:

$$m_{t+1} = \frac{\beta u_1(c^*_{t+1}, l^*_{t+1})}{u_1(c^*_t, l^*_t)}$$
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Recall:

$$E_t m_{t+1} = \frac{1}{1 + r_t^f}$$
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Risk premium on the asset:

$$\frac{E_t m_{t+1} E_t p_{t+1} - E_t m_{t+1} p_{t+1}}{E_t m_{t+1} E_t p_{t+1}} = -\text{Cov}_t(m_{t+1}, p_{t+1})$$
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Lucas-Breeden stochastic discount factor:

$$ m_{t+1} = \frac{\beta u_1(c^*_{t+1}, l^*_{t+1})}{u_1(c^*_t, l^*_t)} $$

Risk premium on the asset:

$$ \frac{E_t m_{t+1} E_t p_{t+1} - E_t m_{t+1} p_{t+1}}{E_t m_{t+1} E_t p_{t+1}} $$

$$ = -\text{Cov}_t(dm_{t+1}, dp_{t+1}) $$

$$ = \frac{-\text{Cov}_t(dm_{t+1}, dp_{t+1})}{E_t m_{t+1} E_t p_{t+1}} $$
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Risk premium on the asset:

$$\frac{E_t m_{t+1} E_t p_{t+1} - E_t m_{t+1} p_{t+1}}{E_t m_{t+1} E_t p_{t+1}}$$

$$= -\text{Cov}_t(dm_{t+1}, dp_{t+1})$$

$$= \frac{E_t m_{t+1}}{E_t m_{t+1}}$$
Asset Pricing

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$$\frac{E_t m_{t+1} E_t p_{t+1} - E_t m_{t+1} p_{t+1}}{E_t m_{t+1} E_t p_{t+1}} 
\approx -\frac{\text{Cov}_t(dm_{t+1}, dp_{t+1})}{\beta}$$
Asset Pricing

\[ m_{t+1} = \frac{\beta u_1(c_{t+1}^*, l_{t+1}^*)}{u_1(c_t^*, l_t^*)} \]
Asset Pricing

\[ dm_{t+1} = \frac{\beta}{u_1(c^*_t, l^*_t)} \left[ u_{11}(c_{t+1}^*, l_{t+1}^*) dc_{t+1}^* + u_{12}(c_{t+1}^*, l_{t+1}^*) dl_{t+1}^* \right]. \]
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Intuitively:

\[ dl^*_{t+1} = -\lambda dc^*_{t+1}, \]

\[ dc^*_{t+1} = \frac{r}{1 + w\lambda} da_{t+1}, \]
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\[ dl_{t+1}^* = -\lambda dc_{t+1}^*, \]

\[ dc_{t+1}^* = \frac{r}{1 + w\lambda} da_{t+1}, \]

So:

\[ dm_{t+1} = \beta \frac{u_{11} - \lambda u_{12}}{u_1} \frac{r}{1 + w\lambda} da_{t+1}. \]
\[ dm_{t+1} = \frac{\beta}{u_1(c_t^*, l_t^*)} \left[ u_{11}(c_{t+1}^*, l_{t+1}^*) dc_{t+1}^* + u_{12}(c_{t+1}^*, l_{t+1}^*) dl_{t+1}^* \right]. \]

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So:
\[ dm_{t+1} = \beta \frac{u_{11} - \lambda u_{12}}{u_1} \frac{r}{1 + w\lambda} da_{t+1}. \]

Risk premium on the asset:
\[ -\text{Cov}_t(dm_{t+1}, dp_{t+1}) \frac{\beta}{\beta} = R^a(a; \theta) \cdot \text{Cov}_t(da_{t+1}, dp_{t+1}). \]
Asset Pricing

\[ dm_{t+1} = \frac{\beta}{u_1(c^*_t, l^*_t)} \left[ u_{11}(c^*_{t+1}, l^*_{t+1}) dc^*_{t+1} + u_{12}(c^*_{t+1}, l^*_{t+1}) dl^*_{t+1} \right]. \]

More carefully:

\[ dl^*_{t+1} = -\lambda dc^*_{t+1} + \frac{u_1}{-u_{22} - wu_{12}} dw_{t+1}, \]

\[ dc^*_{t+1} = \frac{r}{1 + w\lambda} \left[ da_{t+1} + E_{t+1} \sum_{k=1}^{\infty} \frac{1}{(1+r)^k} (ldw_{t+k} + dd_{t+k} + adr_{t+k}) \right] \]

\[ + \frac{u_1 u_{12}}{u_{11} u_{22} - u_{12}^2} dw_{t+1} \]

\[ + \frac{-u_1}{u_{11} - \lambda u_{12}} E_{t+1} \sum_{k=1}^{\infty} \frac{1}{(1+r)^k} \left( \frac{r\lambda}{1 + w\lambda} dw_{t+k} - \beta dr_{t+k+1} \right). \]
\[ dm_{t+1} = \frac{\beta}{u_1(c^*_t, l^*_t)} \left[ u_{11}(c^*_{t+1}, l^*_{t+1}) dc^*_{t+1} + u_{12}(c^*_{t+1}, l^*_{t+1}) dl^*_{t+1} \right]. \]

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\[ + \frac{-u_1}{u_{11} - \lambda u_{12}} E_{t+1} \sum_{k=1}^{\infty} \frac{1}{(1 + r)^k} \left( \frac{r\lambda}{1 + w\lambda} dw_{t+k} - \beta dr_{t+k+1} \right). \]
Asset Pricing

\[
 dm_{t+1} = \frac{\beta}{u_1(c_t^*, l_t^*)} \left[ u_{11}(c_{t+1}^*, l_{t+1}^*) dc_{t+1}^* + u_{12}(c_{t+1}^*, l_{t+1}^*) dl_{t+1}^* \right].
\]

More carefully:

\[
dl_{t+1}^* = -\lambda dc_{t+1}^* + \frac{u_1}{-u_{22} - wu_{12}} dw_{t+1},
\]

\[
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\]

\[+ \frac{u_1 u_{12}}{u_{11} u_{22} - u_{12}^2} dw_{t+1}
\]

\[+ \frac{-u_1}{u_{11} - \lambda u_{12}} E_{t+1} \sum_{k=1}^{\infty} \frac{1}{(1 + r)^k} \left( \frac{r\lambda}{1 + w\lambda} dw_{t+k} - \beta dr_{t+k+1} \right).\]
Asset Pricing

\[ dm_{t+1} = \frac{\beta}{u_1(c_{t+1}^*, l_{t+1}^*)} \left[ u_{11}(c_{t+1}^*, l_{t+1}^*) dc_{t+1} + u_{12}(c_{t+1}^*, l_{t+1}^*) dl_{t+1} \right]. \]

More carefully:

\[ dl_{t+1}^* = -\lambda dc_{t+1} + \frac{u_1}{-u_{22} - w u_{12}} dw_{t+1}, \]

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Risk premium on the asset:

\[ R^a(a; \theta) \cdot \text{Cov}_t(dp_{t+1}, d\hat{A}_{t+1}) + \text{Cov}_t(dp_{t+1}, d\psi_{t+1}), \]
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Numerical results for an equity claim to consumption, $\gamma = 200$:

![Graph showing Coefficient of Relative Risk Aversion (left axis) and Equity Premium (right axis).]
Hansen-Rogerson linear-labor preferences are common:

- Monetary search: Lagos-Wright (2005)
- Investment: Khan-Thomas (2008), Bachmann-Caballero-Engel (2010), Bachmann-Bayer (2009)
Risk Neutrality

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The present paper suggests ways to model risk neutrality that do not require linear utility of consumption.
Empirical Estimates of Risk Aversion

Barsky-Juster-Kimball-Shaprio (1997):

“Suppose that you are the only income earner in the family, and you have a good job guaranteed to give you your current (family) income every year for life. You are given the opportunity to take a new and equally good job, with a 50–50 chance it will double your (family) income and a 50–50 chance that it will cut your (family) income by a third. Would you take the new job?”
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Empirical estimates of risk aversion using methods like these remain valid in the framework of the present paper.

What is different is how these estimates are mapped into model parameters (i.e., risk aversion $\neq -cu_{11}/u_1$)
Empirical Asset Pricing

Campbell (1996, 1999): \[ u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \quad m_{t+1} = \log \beta - \gamma \Delta c_{t+1} \]
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<table>
<thead>
<tr>
<th>Country</th>
<th>( E_t(r_{e,t} - r_{f,t}) )</th>
<th>std(( r_{e,t} - r_{f,t} ))</th>
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If \( u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma} - \eta \frac{L_t^{1+\chi}}{1+\chi} \), then \( \gamma \neq \) risk aversion.
Conclusions

1. The labor margin has dramatic effects on risk aversion

2. Risk aversion is the right concept for asset pricing

3. Arrow-Pratt risk neutrality holds for any \( u \) with \( u_{11}u_{22} - u_{12}^2 = 0 \)

4. Risk aversion and the intertemporal elasticity of substitution are nonreciprocal when there is labor in the model

5. Simple, closed-form expressions for risk aversion in DSGE models with:
   - expected utility preferences
   - Epstein-Zin preferences
   - external or internal habits
   - valid away from steady state