Risk Aversion and the Labor Margin in Dynamic Equilibrium Models

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Abstract

The household's labor margin has substantial effects on risk aversion, and hence asset prices, in dynamic equilibrium models even when utility is additively separable between consumption and labor. This paper derives simple, closed-form expressions for risk aversion that take into account the household's labor margin. Ignoring this margin can dramatically overstate the household's true aversion to risk. Risk premia on assets priced with the stochastic discount factor increase essentially linearly with risk aversion, so measuring risk aversion correctly is crucial for asset pricing in the model. Closed-form expressions for risk aversion in models with generalized recursive preferences and internal and external habits are also derived.

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1. Introduction

In a static, one-period model with household utility $u(\cdot)$ defined over a single consumption good, Arrow (1964) and Pratt (1965) defined the coefficients of absolute and relative risk aversion, $-u''(c)/u'(c)$ and $-c\,u''(c)/u'(c)$. Difficulties immediately arise, however, when one attempts to generalize these concepts to the case of many periods or many goods (e.g., Kihlstrom and Mirman, 1974). These difficulties are particularly pronounced in a dynamic equilibrium model with labor, in which there is a double infinity of goods to consider—consumption and labor in every future period and state of nature—all of which may vary in response to a typical shock to asset returns or wealth.

The present paper shows how to compute risk aversion in dynamic equilibrium models. First, we verify that risk aversion depends on the partial derivatives of the household's value function V with respect to wealth a—that is, the coefficients of absolute and relative risk aversion are essentially $-V_{aa}/V_a$ and $-aV_{aa}/V_a$, respectively. Even though closed-form solutions for the value function do not exist in general, we can derive simple, closed-form expressions for risk aversion at the model's steady state, or along a balanced growth path, by using the fact that the derivative of the value function with respect to wealth equals the current-period marginal utility of consumption (Benveniste and Scheinkman, 1979). Importantly, these closed-form expressions for risk aversion seem to remain good approximations even far away from the model's steady state.

A main result of the paper is that the household's labor margin has substantial effects on risk aversion, and hence asset prices. Even when labor and consumption are additively separable in utility, they remain connected by the household's budget constraint: in particular, the household can absorb shocks to asset returns either through changes in consumption, changes in hours worked, or some combination of the two. This ability to absorb shocks along either or both margins greatly alters the household's attitudes toward risk. For example, if the household's utility kernel is given by $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t$, the quantity $-c\,u_{11}/u_1 = \gamma$ is often referred to as the household's coefficient of relative risk aversion, but in fact the household is *risk neutral* with respect to gambles over asset values or wealth. Intuitively, the household is indifferent at the margin between using labor or consumption to absorb a shock to assets, and the household in this example is clearly risk neutral with respect to gambles over hours. More generally, when $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$, risk aversion equals $(\gamma^{-1} + \chi^{-1})^{-1}$, a combination of the parameters on the household's consumption and labor margins, reflecting that the household absorbs shocks using both margins.[1]

While modeling risk neutrality is not a main goal of the present paper, risk neutrality nevertheless can be a desirable feature for some applications, such as labor market search or financial frictions, since it allows for closed-form solutions to key features of the model.[2] Thus, an additional contribution of the present paper is to show ways of modeling risk neutrality that do not require utility to be linear in consumption, which has undesirable implications for interest rates and consumption growth. Instead, any utility kernel with a singular Hessian matrix can be used.

A final result of the paper is that risk premia computed using the Lucas-Breeden stochastic discounting framework are essentially linear in risk aversion. That is, measuring risk aversion correctly—taking into account the household's labor margin—is necessary for understanding asset prices in the model. Since much recent research has focused on bringing dynamic stochastic general equilibrium (DSGE) models into closer agreement with asset prices,[3] it is surprising that so little attention has been paid to measuring risk aversion correctly in these models. The present paper aims to fill that void.

Closed-form expressions and extensions of the above results to dynamic equilibrium models with generalized recursive preferences (e.g., Epstein and Zin, 1989, Weil, 1989) and habits (e.g., Campbell and Cochrane, 1999) are also derived.

There are a few previous studies that extend the Arrow-Pratt definition beyond the one-good, one-period case. In a static, multiple-good setting, Stiglitz (1969) measures risk aversion using the household's indirect utility function rather than utility itself, essentially a special case of Proposition 1 of the present paper. Constantinides (1990) measures risk aversion in a dynamic, consumption-only (endowment) economy using the household's value function, another special case of Proposition 1. Boldrin, Christiano, and Fisher (1997) apply Constantinides' definition to some very simple endowment economy models for which they

[1] The intertemporal elasticity of substitution in this example is $1/\gamma$, so a corollary of this result is that risk aversion and the intertemporal elasticity of substitution are nonreciprocal when labor supply can vary.

[2] See, e.g., Mortensen and Pissarides (1994), and Bernanke, Gertler, and Gilchrist (1999).

[3] See, e.g., Jermann (1998), Boldrin, Christiano, and Fisher (2001), Tallarini (2000), Rudebusch and Swanson (2008, 2009), Uhlig (2007), Van Binsbergen, Fernandez-Villaverde, Koijen, and Rubio-Ramirez (2010), and Backus, Routledge, and Zin (2009).

can compute closed-form expressions for the value function, and hence risk aversion. The present paper builds on these studies by deriving closed-form solutions for risk aversion in dynamic equilibrium models in general, demonstrating the importance of the labor margin, and showing the tight link between risk aversion and asset prices in these models.

The remainder of the paper proceeds as follows. Section 2 defines the dynamic equilibrium framework within which we study risk aversion. Section 3 presents the main ideas of the paper, deriving risk aversion in dynamic equilibrium models for the time-separable expected utility case. Section 4 demonstrates the close connection between risk aversion and asset pricing in the Lucas-Breeden framework. Section 5 extends the analysis to the case of generalized recursive preferences (Epstein and Zin, 1989), which have been the focus of much recent research at the boundary between macroeconomics and finance. Section 6 extends the analysis to the case of internal and external habits, two of the most common intertemporal nonseperabilities in preferences in both the macroeconomics and finance literatures. Section 7 solves for risk aversion numerically and shows the accuracy of the closed-form expressions. Section 8 discusses some general implications and concludes. An Appendix provides details of proofs and computations that are outlined in the main text.

2. Dynamic Equilibrium Framework

2.1 The Household's Optimization Problem and Value Function

Time is discrete and continues forever. At each time t, the household seeks to maximize the expected present discounted value of utility flows:

$$E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_\tau, l_\tau), \tag{1}$$

subject to the sequence of asset accumulation equations:

$$a_{\tau+1} = (1+r_\tau)a_\tau + w_\tau l_\tau + d_\tau - c_\tau, \qquad \tau = t, t+1, \dots \tag{2}$$

and the no-Ponzi-scheme condition:

$$\lim_{T\to\infty} \prod_{\tau=t}^{T} (1+r_{\tau+1})^{-1} a_{T+1} \geq 0, \tag{3}$$

where E_t denotes the mathematical expectation conditional on the household's information set at time t, $\beta \in (0,1)$ is the household's discount factor, $(c_t, l_t) \in \Omega \subseteq \mathbb{R}^2$ denotes the household's choice of consumption and labor in period t, a_t is the household's beginning-of-period assets, and w_t, r_t, and d_t denote the real wage, interest rate, and net transfer payments at time t. There is a finite-dimensional Markovian state vector θ_t that is exogenous to the household and governs the processes for w_t, r_t, and d_t. Conditional on θ_t, the household knows the time-t values for w_t, r_t, and d_t. The state vector and information set of the household's optimization problem at each date t is thus $(a_t; \theta_t)$, and we denote the domain of $(a_t; \theta_t)$ by X. Let $\Gamma : X \to \Omega$ describe the set-valued correspondence of feasible choices for (c_t, l_t) for each given $(a_t; \theta_t)$.

We make the following regularity assumptions regarding the utility kernel u:

Assumption 1. *The function $u : \Omega \to \mathbb{R}$ is increasing in its first argument, decreasing in its second, twice-differentiable, and strictly concave.*

In addition to Assumption 1, a few more technical conditions are required to ensure the value function for the household's optimization problem exists and satisfies the Bellman equation (Stokey and Lucas (1990), Rincón-Zapatera and Rodríguez-Palmero (2003), and Marinacci and Montrucchio (2010) give different sets of such sufficient conditions). The details of these conditions are tangential to the present paper, so we simply assume:

Assumption 2. *The value function $V : X \to \mathbb{R}$ for the household's optimization problem exists and satisfies the Bellman equation:*

$$V(a_t; \theta_t) \;=\; \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} \; u(c_t, l_t) + \beta E_t V(a_{t+1}; \theta_{t+1}), \tag{4}$$

where a_{t+1} is given by (2).

Together, Assumptions 1–2 guarantee the existence of a unique optimal choice for (c_t, l_t) at each point in time, given $(a_t; \theta_t)$. Let $c_t^* \equiv c^*(a_t; \theta_t)$ and $l_t^* \equiv l^*(a_t; \theta_t)$ denote the household's optimal choices of c_t and l_t as functions of the state $(a_t; \theta_t)$. Then V can be written as:

$$V(a_t; \theta_t) = u(c_t^*, l_t^*) + \beta E_t V(a_{t+1}^*; \theta_{t+1}), \tag{5}$$

where $a_{t+1}^* \equiv (1 + r_t)a_t + w_t l_t^* + d_t - c_t^*$.

To avoid boundary solutions, we make the following standard assumption:

Assumption 3. *For any* $(a_t;\theta_t) \in X$*, the household's optimal choice* (c_t^*, l_t^*) *lies in the interior of* $\Gamma(a_t;\theta_t)$.

Intuitively, Assumption 3 requires the partial derivatives of u to grow sufficiently large toward the boundary that only interior solutions for c_t^* and l_t^* are optimal for all $(a_t;\theta_t) \in X$.

Assumptions 1–3 guarantee that V is continuously differentiable and satisfies the Benveniste-Scheinkman equation, but we will require slightly more than this below:

Assumption 4. *The value function* $V(\cdot;\cdot)$ *is twice-differentiable.*

Assumption 4 also implies differentiability of the optimal policy functions, c^* and l^*. Santos (1991) provides relatively mild sufficient conditions for Assumption 4 to be satisfied; intuitively, u must be strongly concave.

2.2 Representative Household and Steady State Assumptions

Up to this point, we have considered the case of a single household in isolation, leaving the other households of the model and the production side of the economy unspecified. Implicitly, the other households and production sector jointly determine the process for θ_t (and hence w_t, r_t, and d_t), and much of the analysis below does not need to be any more specific about these processes than this. However, to move from general expressions for risk aversion to more concrete, closed-form expressions, we adopt two standard assumptions from the DSGE literature.[4]

Assumption 5. *The household is* atomistic *and* representative.

Assumption 5 implies that the individual household's choices for c_t and l_t have no effect on the aggregate quantities w_t, r_t, d_t, and θ_t. It also implies that, when the economy is at the nonstochastic steady state (described shortly), any individual household finds it optimal to choose the steady-state values of c and l given a and θ.

Assumption 6. *The model has a nonstochastic steady state, or a balanced growth path that can be renormalized to a nonstochastic steady state after a suitable change of variables. At the nonstochastic steady state,* $x_t = x_{t+1} = x_{t+k}$ *for* $k = 1, 2, \ldots,$ *and* $x \in \{c, l, a, w, r, d, \theta\}$*, and we drop the subscript* t *to denote the steady-state value.*

[4] Alternative assumptions about the nature of the other households in the model or the production sector may also allow for closed-form expressions for risk aversion. However, the assumptions used here are standard and thus the most natural to pursue.

It is important to note that Assumptions 5–6 do not prohibit us from offering an individual household a hypothetical gamble of the type described below. The steady state of the model serves only as a reference point around which the *aggregate* variables w, r, d, and θ and the *other households'* choices of c, l, and a can be predicted with certainty. This reference point is important because it is there that we can compute closed-form expressions for risk aversion.

3. Risk Aversion

3.1 The Coefficient of Absolute Risk Aversion

The household's risk aversion at time t generally depends on the household's state vector at time t, $(a_t; \theta_t)$. Given this state, we consider the household's aversion to a hypothetical one-shot gamble in period t of the form:

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t + \sigma\varepsilon_{t+1}, \tag{6}$$

where ε_{t+1} is a random variable representing the gamble, with bounded support $[\underline{\varepsilon}, \overline{\varepsilon}]$, mean zero, unit variance, independent of θ_τ for all τ, and independent of a_τ, c_τ, and l_τ for all $\tau \leq t$. A few words about (6) are in order: First, the gamble is dated $t+1$ to clarify that its outcome is not in the household's information set at time t. Second, c_t cannot be made the subject of the gamble without substantial modifications to the household's optimization problem, because c_t is a choice variable under control of the household at time t. However, (6) is clearly equivalent to a one-shot gamble over net transfers d_t or asset returns r_t, both of which are exogenous to the household. Indeed, thinking of the gamble as being over r_t helps to illuminate the connection between (6) and the price of risky assets, to which we will return in Section 4. As shown there, the household's aversion to the gamble in (6) is directly linked to the premium households require to hold risky assets.

Following Arrow (1964) and Pratt (1965), we can ask what one-time fee μ the household would be willing to pay in period t to avoid the gamble in (6):

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t - \mu. \tag{7}$$

The quantity μ that makes the household just indifferent between (6) and (7), for infinitesimal

μ and σ, is the household's coefficient of absolute risk aversion:[5]

Definition 1. *Let* $(a_t; \theta_t)$ *be an interior point of* X*, let* $\widetilde{V}(a_t; \theta_t; \sigma)$ *denote the value function for the household's optimization problem inclusive of the one-shot gamble* (6)*, and let* $\mu(\sigma)$ *denote the value of* μ *that satisfies* $V(a_t - \frac{\mu}{1+r_t}; \theta_t) = \widetilde{V}(a_t; \theta_t; \sigma)$*. The household's coefficient of absolute risk aversion at* $(a_t; \theta_t)$ *is given by* $\lim_{\sigma \to 0} 2\mu(\sigma)/\sigma^2$*.*

The following proposition verifies that the coefficient of absolute risk aversion is well-defined and equals the "folk wisdom" value of $-V_{11}/V_1$:[6]

Proposition 1. *Let* $(a_t; \theta_t)$ *be an interior point of* X*. The household's coefficient of absolute risk aversion at* $(a_t; \theta_t)$ *exists and equals:*

$$\frac{-\, E_t V_{11}(a^*_{t+1}; \theta_{t+1})}{E_t V_1(a^*_{t+1}; \theta_{t+1})}\,, \tag{8}$$

where V_1 *and* V_{11} *denote the first and second partial derivatives of* V *with respect to its first argument. Evaluated at the steady state,* (8) *simplifies to:*

$$\frac{-V_{11}(a; \theta)}{V_1(a; \theta)}\,. \tag{9}$$

Proof: See Appendix.

Equations (8)–(9) are essentially Constantinides' (1990) definition of risk aversion, and have obvious similarities to Arrow (1964) and Pratt (1965). Here, of course, it is the curvature of the value function V with respect to assets that matters, rather than the curvature of the utility kernel u with respect to consumption.[7]

Deriving the coefficient of absolute risk aversion in Proposition 1 is simple enough, but the problem with (8)–(9) is that closed-form expressions for V (and hence V_1 and V_{11}) do not exist in general, even for the simplest dynamic models with labor. This difficulty may help to explain the popularity of "shortcut" approaches to measuring risk aversion, notably $-u_{11}(c^*_t, l^*_t)/\, u_1(c^*_t, l^*_t)$, which has no clear relationship to (8)–(9) except in the one-good one-period case. Boldrin, Christiano, and Fisher (1997) derive closed-form solutions for V—and

[5] We defer discussion of relative risk aversion until the next subsection because defining total household wealth is complicated by the presence of human capital—that is, the household's labor income.

[6] See, e.g., Constantinides (1990), Farmer (1990), Cochrane (2001), and Flavin and Nakagawa (2008). One contribution of the present paper is to prove this folk wisdom rigorously, although the paper's contributions extend beyond just Proposition 1.

[7] Arrow (1964) and Pratt (1965) occasionally refer to utility as being defined over "money", so one could argue that they always intended for risk aversion to be measured using indirect utility or the value function.

hence risk aversion—for some very simple, consumption-only endowment economy models. This approach is a nonstarter for even the simplest dynamic models with labor.

We solve this problem by observing that V_1 and V_{11} often can be computed even when closed-form solutions for V cannot be. For example, the Benveniste-Scheinkman equation:

$$V_1(a_t;\theta_t) = (1+r_t)\, u_1(c_t^*, l_t^*), \tag{10}$$

states that the marginal value of a dollar of assets equals the marginal utility of consumption times $1+r_t$ (the interest rate appears here because beginning-of-period assets in the model generate income in period t). In (10), u_1 is a known function. Although closed-form solutions for the functions c^* and l^* are not known in general, the points c_t^* and l_t^* often are known—for example, when they are evaluated at the nonstochastic steady state, c and l. Thus, we can compute V_1 at the nonstochastic steady state by evaluating (10) at that point.

We compute V_{11} by noting that (10) holds for general a_t; hence we can differentiate (10) to yield:

$$V_{11}(a_t;\theta_t) \;=\; (1+r_t)\left[u_{11}(c_t^*, l_t^*)\,\frac{\partial c_t^*}{\partial a_t} \;+\; u_{12}(c_t^*, l_t^*)\,\frac{\partial l_t^*}{\partial a_t}\right]. \tag{11}$$

All that remains is to find the derivatives $\partial c_t^*/\partial a_t$ and $\partial l_t^*/\partial a_t$.

We solve for $\partial l_t^*/\partial a_t$ by differentiating the household's intratemporal optimality condition:

$$-u_2(c_t^*, l_t^*) \;=\; w_t\, u_1(c_t^*, l_t^*), \tag{12}$$

with respect to a_t, and rearranging terms to yield:

$$\frac{\partial l_t^*}{\partial a_t} = -\lambda_t \frac{\partial c_t^*}{\partial a_t}\,, \tag{13}$$

where

$$\lambda_t \;\equiv\; \frac{w_t u_{11}(c_t^*, l_t^*) + u_{12}(c_t^*, l_t^*)}{u_{22}(c_t^*, l_t^*) + w_t u_{12}(c_t^*, l_t^*)} \;=\; \frac{u_1(c_t^*, l_t^*)u_{12}(c_t^*, l_t^*) - u_2(c_t^*, l_t^*)u_{11}(c_t^*, l_t^*)}{u_1(c_t^*, l_t^*)u_{22}(c_t^*, l_t^*) - u_2(c_t^*, l_t^*)u_{12}(c_t^*, l_t^*)}\,. \tag{14}$$

Note that, if consumption and leisure in period t are normal goods, then $\lambda_t > 0$, although we do not require this restriction below. It now only remains to solve for the derivative $\partial c_t^*/\partial a_t$.

Intuitively, $\partial c_t^*/\partial a_t$ should not be too difficult to compute: it is just the household's marginal propensity to consume today out of a change in assets, which we can deduce from the household's Euler equation and budget constraint. Differentiating the Euler equation:

$$u_1(c_t^*, l_t^*) \;=\; \beta E_t(1+r_{t+1})\, u_1(c_{t+1}^*, l_{t+1}^*), \tag{15}$$

with respect to a_t yields:[8]

$$u_{11}(c_t^*, l_t^*)\frac{\partial c_t^*}{\partial a_t} + u_{12}(c_t^*, l_t^*)\frac{\partial l_t^*}{\partial a_t} = \beta E_t(1+r_{t+1})\left[u_{11}(c_{t+1}^*, l_{t+1}^*)\frac{\partial c_{t+1}^*}{\partial a_t} + u_{12}(c_{t+1}^*, l_{t+1}^*)\frac{\partial l_{t+1}^*}{\partial a_t}\right] \tag{16}$$

Substituting in for $\partial l_t^*/\partial a_t$ gives:

$$(u_{11}(c_t^*, l_t^*) - \lambda_t u_{12}(c_t^*, l_t^*))\frac{\partial c_t^*}{\partial a_t} = \beta E_t(1+r_{t+1})\,(u_{11}(c_{t+1}^*, l_{t+1}^*) - \lambda_{t+1} u_{12}(c_{t+1}^*, l_{t+1}^*))\frac{\partial c_{t+1}^*}{\partial a_t}\,. \tag{17}$$

Evaluating (17) at steady state, $\beta = (1+r)^{-1}$, $\lambda_t = \lambda_{t+1} = \lambda$, and the u_{ij} cancel, giving:

$$\frac{\partial c_t^*}{\partial a_t} = E_t\frac{\partial c_{t+1}^*}{\partial a_t} = E_t\frac{\partial c_{t+k}^*}{\partial a_t}\,, \quad k = 1, 2, \ldots \tag{18}$$

$$\frac{\partial l_t^*}{\partial a_t} = E_t\frac{\partial l_{t+1}^*}{\partial a_t} = E_t\frac{\partial l_{t+k}^*}{\partial a_t}\,, \quad k = 1, 2, \ldots \tag{19}$$

In other words, whatever the change in the household's consumption today, it must be the same as the expected change in consumption tomorrow, and the expected change in consumption at each future date $t+k$.[9]

The household's budget constraint is implied by asset accumulation equation (2) and transversality condition (3). Differentiating (2) with respect to a_t, evaluating at steady state, and applying (3), (18), and (19) gives:

$$\frac{1+r}{r}\frac{\partial c_t^*}{\partial a_t} = (1+r) + \frac{1+r}{r}\,w\,\frac{\partial l_t^*}{\partial a_t}\,. \tag{20}$$

That is, the expected present value of changes in household consumption must equal the change in assets (times $1+r$) plus the expected present value of changes in labor income.

Combining (20) with (13), we can solve for $\partial c_t^*/\partial a_t$ evaluated at the steady state:

$$\frac{\partial c_t^*}{\partial a_t} = \frac{r}{1+w\lambda}\,. \tag{21}$$

In response to a unit increase in assets, the household raises consumption in every period by the extra asset income r (the "golden rule"), adjusted downward by the amount $1+w\lambda$ that takes into account the household's decrease in labor income.

[8] By $\partial c_{t+1}^*/\partial a_t$ we mean:

$$\frac{\partial c_{t+1}^*}{\partial a_t} = \frac{\partial c_{t+1}^*}{\partial a_{t+1}}\frac{da_{t+1}^*}{da_t} = \frac{\partial c_{t+1}^*}{\partial a_{t+1}}\left[1 + r_{t+1} + w_t\frac{\partial l_t^*}{\partial a_t} - \frac{\partial c_t^*}{\partial a_t}\right],$$

and analogously for $\partial l_{t+1}^*/\partial a_t$, $\partial c_{t+2}^*/\partial a_t$, $\partial l_{t+2}^*/\partial a_t$, etc.

[9] Note that this equality does not follow from the steady state assumption. For example, in a model with internal habits, which we will consider in Section 6, the individual household's optimal consumption response to a change in assets increases with time, even starting from steady state.

We can now compute the household's coefficient of absolute risk aversion. Substituting (10), (11), (13)–(14), and (21) into (9), we have proved:

Proposition 2. *The household's coefficient of absolute risk aversion in Proposition 1, evaluated at steady state, satisfies:*

$$\frac{-V_{11}(a;\theta)}{V_1(a;\theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{r}{1 + w\lambda}, \tag{22}$$

where u_1*,* u_{11}*, and* u_{12} *denote the corresponding partial derivatives of* u *evaluated at the steady state* (c, l)*, and* λ *is given by* (14) *evaluated at steady state.*

When there is no labor margin in the model, Proposition 2 has the following corollary:

Corollary 3. *Suppose that* l_t *is fixed exogenously at some* $\bar{l} \in \mathbb{R}$ *for all* t *and the household chooses* c_t *optimally at each* t *given this constraint. Then the household's coefficient of absolute risk aversion* (22)*, evaluated at steady state, is given by:*

$$\frac{-V_{11}(a;\theta)}{V_1(a;\theta)} = \frac{-u_{11}}{u_1} r. \tag{23}$$

Proof: The assumptions and steps leading up to Proposition 2, adjusted to the one-dimensional case, are essentially the same as the above with $\lambda_t = 0$.

Proposition 2 and Corollary 3 are remarkable. First, the household's coefficient of absolute risk aversion in (23) is just the traditional measure, $-u_{11}/u_1$, times r, which translates assets into current-period consumption. In other words, for any utility kernel u, the traditional, static measure of risk aversion is also the correct measure in the dynamic context, regardless of whether or not u is homothetic or the rest of the model is homogeneous, whether or not we can solve for V, and no matter what the functional forms of u and V.

More generally, when households have a labor margin, Proposition 2 shows that risk aversion is less than the traditional measure by the factor $1 + w\lambda$, *even when consumption and labor are additively separable in* u (i.e., $u_{12} = 0$). Even in the additively separable case, households can partially absorb shocks to income through changes in hours worked. As a result, c_t^* depends on household labor supply, so labor and consumption are indirectly connected through the budget constraint.[10] When $u_{12} \neq 0$, risk aversion in Proposition 2 is

[10] Uhlig (2007) notes that, if households have Epstein-Zin preferences, then leisure must be taken into account in pricing assets because the value function V appears in the stochastic discount factor, and V depends on leisure. The present paper makes the point that the labor margin affects asset prices even in the case of additively separable expected utility preferences, because the labor margin changes the household's consumption process. The present paper also derives closed-form expressions for risk aversion, relates them to asset prices, and shows that those expression remain good approximations away from the steady state.

further attenuated or amplified by the direct interaction between consumption and labor in utility, u_{12}. Note, however, that regardless of the signs of λ and u_{12}, risk aversion is always reduced, on net, when households can vary their labor supply:

Corollary 4. *The coefficient of absolute risk aversion* (22) *is less than or equal to* (23),

$$\frac{-u_{11}+\lambda u_{12}}{u_1}\,\frac{r}{1+w\lambda} \;\leq\; \frac{-u_{11}}{u_1}\,r\,. \tag{24}$$

If $r<1$*, then* (22) *is also less than* $-u_{11}/u_1$*.*

Proof: Substituting in for λ and w, (22) can be written as:

$$\frac{-ru_{11}}{u_1}\,\frac{u_{11}u_{22}-u_{12}^2}{u_{11}u_{22}-2\frac{u_2}{u_1}\,u_{11}u_{12}+\left(\frac{u_2}{u_1}\right)^2u_{11}^2} \;=\; \frac{-ru_{11}}{u_1}\,\frac{1}{1+\dfrac{\left(\frac{u_2}{u_1}\,u_{11}-u_{12}\right)^2}{u_{11}u_{22}-u_{12}^2}}\,. \tag{25}$$

Strict concavity of u implies that $u_{11}u_{22}-u_{12}^2>0$, hence the right-hand side of (25) is less than or equal to $-ru_{11}/u_1$.

Since r denotes the net interest rate, $r \ll 1$ in typical calibrations, satisfying the condition at the end of Corollary 4.

The household's labor margin can have dramatic effects on risk aversion. For example, from the left-hand side of (25) it is apparent that, no matter how large is $-u_{11}/u_1$, risk aversion can be arbitrarily small as the matrix discriminant, $u_{11}u_{22}-u_{12}^2$, approaches zero.[11] In other words, risk aversion depends on the concavity of u in all dimensions rather than just in one dimension. Even when u_{11} is very large, the household still can be risk neutral if u_{22} is small or the cross-effect u_{12} is sufficiently large. Geometrically, if there exists any direction in (c,l)-space along which u is flat, the household will optimally choose to absorb shocks to income along that line, resulting in risk-neutral behavior.

We provide some more concrete examples of risk aversion calculations in Section 3.3, below, after first defining relative risk aversion.

[11] In constructing examples, one must also ensure that the denominator, $u_{11}u_{22}-2(u_2/u_1)u_{11}u_{12}+(u_2/u_1)^2u_{11}^2$, does not vanish, which amounts to ensuring that $[-u_2,u_1]'$ is not in the null space of the Hessian of u. An example of this type was provided in the Introduction, where $u_{22}=u_{12}=0$ and $u_2\neq 0$. See also the discussion of Examples 1 and 2, below.

3.2 The Coefficient of Relative Risk Aversion

The difference between absolute and relative risk aversion is the size of the hypothetical gamble faced by the household. If the household faces a one-shot gamble of size A_t in period t, that is:

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t + A_t \sigma \varepsilon_{t+1}, \tag{26}$$

or the household can pay a one-time fee $A_t \mu$ in period t to avoid this gamble, then it follows from Proposition 1 that the household's coefficient of risk aversion, $\lim_{\sigma \to 0} 2\mu(\sigma)/\sigma^2$, for this gamble is given by:

$$\frac{-A_t E_t V_{11}(a^*_{t+1}; \theta_{t+1})}{E_t V_1(a^*_{t+1}; \theta_{t+1})}. \tag{27}$$

The natural definition of A_t, considered by Arrow (1964) and Pratt (1965), is the household's wealth at time t. The gamble in (26) is then over a fraction of the household's wealth and (27) is referred to as the household's coefficient of relative risk aversion.

In models with labor, however, household wealth can be more difficult to define because of the presence of human capital. In these models, there are two natural definitions of human capital, so we consequently define two measures of household wealth A_t and two coefficients of relative risk aversion (27).

First, when the household's time endowment is not well-defined—as when $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}$ and no upper bound $\bar{l}$ on l_t is specified, or $\bar{l}$ is specified but is completely arbitrary—it is most natural to define human capital as the present discounted value of labor income, $w_t l^*_t$. Equivalently, total household wealth A_t equals the present discounted value of consumption, which follows from the budget constraint (2)–(3). We state this formally as:

Definition 2. *The household's consumption-based coefficient of relative risk aversion is given by* (27)*, with* $A_t \equiv (1+r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau} c^*_\tau$*, the present discounted value of household consumption, and where* $m_{t,\tau}$ *denotes the stochastic discount factor* $\beta^{\tau-t} u_1(c^*_\tau, l^*_\tau)/u_1(c^*_t, l^*_t)$.

The factor $(1 + r_t)^{-1}$ in the definition expresses wealth A_t in beginning- rather than end-of-period-t units, so that in steady state $A = c/r$ and the consumption-based coefficient of relative risk aversion is given by:

$$\frac{-A\, V_{11}(a; \theta)}{V_1(a; \theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1 + w\lambda}. \tag{28}$$

Alternatively, when the household's time endowment $\bar{l}$ is well specified, we can define human capital to be the present discounted value of the household's time endowment, $w_t\bar{l}$. In thise case, total household wealth $\tilde{A}_t$ equals the present discounted value of leisure $w_t(\bar{l}-l_t^*)$ plus consumption c_t^*, from (2)–(3). We thus have:

Definition 3. *The household's leisure-and-consumption-based coefficient of relative risk aversion is given by* (27)*, with* $A_t = \tilde{A}_t \equiv (1+r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau}\big(c_\tau^* + w_\tau(\bar{l} - l_\tau^*)\big)$.

In steady state, $\tilde{A} = \big(c + w(\bar{l} - l)\big)/r$, and the leisure-and-consumption-based coefficient of relative risk aversion is given by:

$$\frac{-\tilde{A}\, V_{11}(a;\theta)}{V_1(a;\theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \, \frac{c + w(\bar{l} - l)}{1 + w\lambda} \,. \tag{29}$$

Of course, (28) and (29) are related by the ratio of the two gambles, $(c + w(\bar{l} - l))/c$.

Other definitions of relative risk aversion, corresponding to alternative definitions of wealth and the size of the gamble A_t, are also possible, but Definitions 2–3 are the most natural for several reasons. First, both definitions reduce to the usual present discounted value of income or consumption when there is no human capital in the model. Second, both measures of risk aversion reduce to the traditional $-c\,u_{11}/u_1$ when there is no labor margin in the model—that is, when $\lambda = 0$. Third, in steady state the household consumes exactly the flow of income from its wealth, rA, consistent with standard permanent income theory (where one must include the value of leisure $w(\bar{l} - l)$ as part of consumption when the value of leisure is included in wealth).

We close this section by noting that neither measure of relative risk aversion is reciprocal to the intertemporal elasticity of substitution:

Corollary 5. *Evaluated at steady state: i) the consumption-based coefficient of relative risk aversion and intertemporal elasticity of substitution are reciprocal if and only if* $\lambda = 0$*; ii) the leisure-and-consumption-based coefficient of relative risk aversion and intertemporal elasticity of substitution are reciprocal if and only if* $\lambda = (\bar{l} - l)/c$.

Proof: The case $w = 0$ is ruled out by Assumptions 1 and 3. The intertemporal elasticity of substitution, evaluated at steady state, is given by $\big(dc_{t+1}^* - dc_t^*\big)/d\log(1 + r_{t+1})$, which equals $-u_1/\big(c(u_{11} - \lambda u_{12})\big)$ by a calculation along the lines of (17), holding w_t fixed but allowing l_t^* and l_{t+1}^* to vary endogenously. The corollary then follows by comparison to (28) and (29).

3.3 Examples

Some simple examples illustrate how ignoring the household's labor margin can lead to extremely inaccurate measures of the household's true attitudes toward risk.

Example 1. Consider the additively separable utility kernel:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}\,, \tag{30}$$

where $\gamma,\ \chi,\ \eta > 0$. The traditional measure of risk aversion for this utility kernel is $-c\,u_{11}/u_1 = \gamma$, but the household's consumption-based coefficient of relative risk aversion is given by (28):

$$\frac{-AV_{11}}{V_1} = \frac{-cu_{11}}{u_1} \frac{1}{1 + w\,\frac{wu_{11}}{u_{22}}} = \frac{\gamma}{1 + \frac{\gamma}{\chi}\frac{wl}{c}}\,. \tag{31}$$

The household's leisure-and-consumption-based coefficient of relative risk aversion (29) is not well defined in this example (the household's risk aversion can be made arbitrarily large or small just by varying the household's time endowment $\bar{l}$), so we focus only on the consumption-based measure (31).

In steady state, $c \approx wl$,[12] so (31) can be written as:

$$\frac{-AV_{11}}{V_1} \approx \frac{1}{\frac{1}{\gamma} + \frac{1}{\chi}}\,. \tag{32}$$

Note that (32) is less than the traditional measure of risk aversion by a factor of $1 + \gamma/\chi$. Thus, if $\gamma = 2$ and $\chi = 1$—parameter values that are well within the range of estimates in the literature—then the household's true risk aversion is less than the traditional measure by a factor of about three. This point is illustrated in Figure 1, which graphs the coefficient of relative risk aversion for this example as a function of the traditional measure, γ, for several different values of χ. If χ is very large, then the bias from using the traditional measure is small because household labor supply is essentially fixed.[13] However, as χ approaches 0, a common benchmark in the literature, the bias explodes and true risk aversion approaches zero—the household becomes risk neutral. Intuitively, households with linear disutility of

[12] In steady state, $c = ra + wl + d$, so $c = wl$ holds exactly if there is neither capital nor transfers in the model. In any case, $ra + d$ is typically small for standard calibrations in the literature.

[13] Similarly, if γ is very small, the bias from using the traditional measure is small because the household chooses to absorb income shocks almost entirely along its consumption margin. As a result, the labor margin is again almost inoperative.

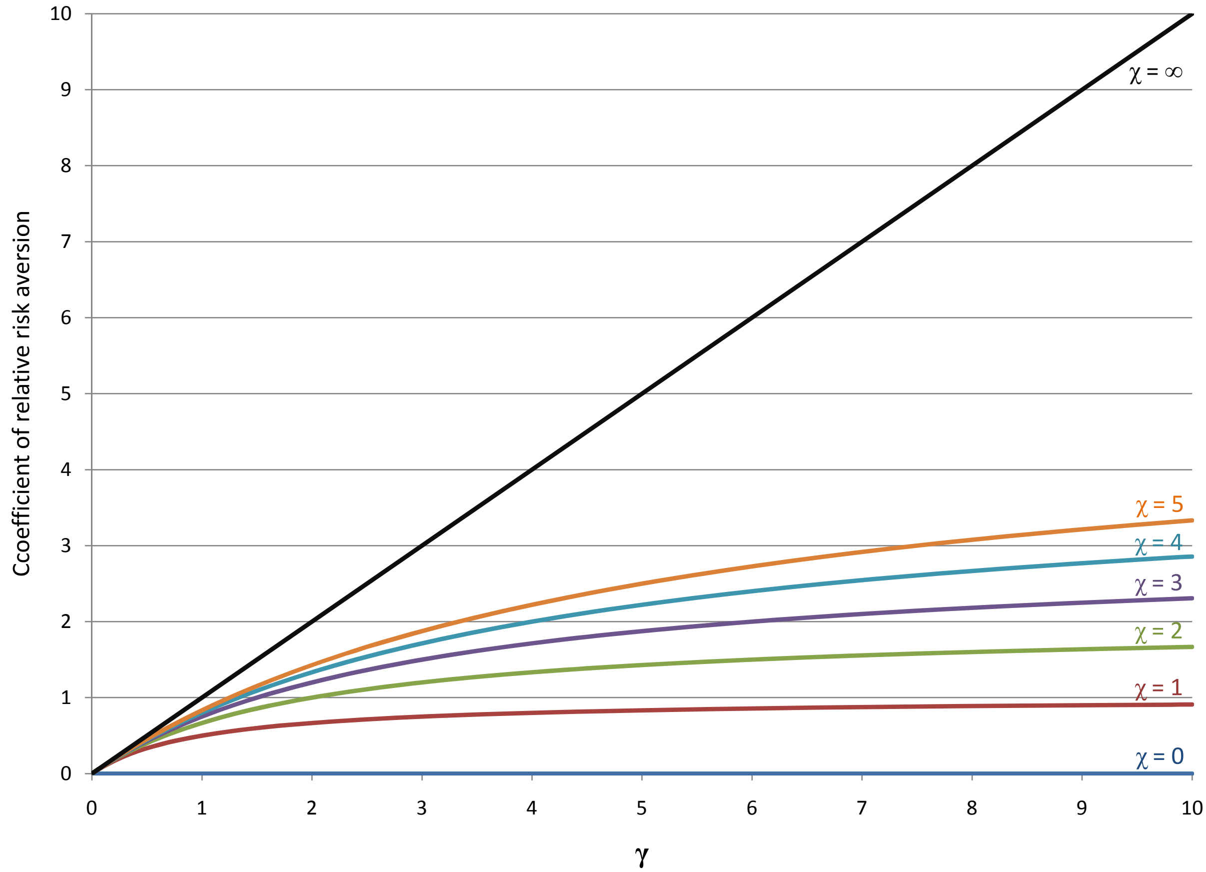

Figure 1. Consumption-based coefficient of relative risk aversion for the utility kernel $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$ in Example 1, as a function of the traditional measure γ, for different values of χ. See text for details.

work are risk neutral with respect to gambles over wealth because they can completely offset those gambles at the margin by working more or fewer hours, and households with linear disutility of work are clearly risk neutral with respect to gambles over hours.

Example 2. Consider the King-Plosser-Rebelo-type (1988) utility kernel:

$$u(c_t, l_t) \;=\; \frac{c_t^{1-\gamma}\,(1-l_t)^{\chi(1-\gamma)}}{1-\gamma}, \tag{33}$$

where $\gamma > 0$, $\gamma \neq 1$, $\chi > 0$, $\bar{l} = 1$, and $\chi(1-\gamma) < \gamma$ for concavity. The traditional measure of risk aversion for (33) is γ, but the household's actual leisure-and-consumption-based coefficient of relative risk aversion is given by:

$$\frac{-\tilde{A}\,V_{11}}{V_1} \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1}\,\frac{c + w(1-l)}{1 + w\lambda} \;=\; \gamma - \chi(1-\gamma). \tag{34}$$

Note that concavity of (33) implies that (34) is positive. As in the previous example, (34) depends on both γ and χ. As χ approaches $\gamma/(1-\gamma)$—that is, as utility approaches Cobb-

Douglas—the household becomes risk neutral; in this case, household utility along the line $c_t = w_t(1 - l_t)$ is linear, so the household finds it optimal to absorb shocks to wealth along that line.

The household's consumption-based coefficient of relative risk aversion is a bit more complicated than (34):

$$\frac{-A\,V_{11}}{V_1} \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1}\,\frac{c}{1 + w\lambda} \;=\; \frac{\gamma - \chi(1-\gamma)}{1+\chi}\,. \tag{35}$$

Neither (34) nor (35) equals the traditional measure γ, except for the special case $\chi = 0$.

It is also worth noting that if

$$u(c_t, l_t) \;=\; \frac{\left(c_t^{1-\chi}\,(1-l_t)^{\chi}\right)^{1-\gamma}}{1-\gamma}\,, \tag{36}$$

$\chi \in (0,1)$, then the consumption-based coefficient of relative risk aversion is γ, the same as if we regard consumption and leisure as a single, composite good.

4. Risk Aversion and Asset Pricing

In the preceding sections, we showed that the labor margin has important implications for Arrow-Pratt risk aversion with respect to gambles over income or wealth. We now show that risk aversion with respect to these gambles is the right concept for asset pricing.

4.1 Measuring Risk Aversion with V As Opposed to u

Some comparison of the expressions $-V_{11}/V_1$ and $-u_{11}/u_1$ helps to clarify why the former measure is the relevant one for pricing assets, such as stocks or bonds, in the model. From Proposition 1, $-V_{11}/V_1$ is the Arrow-Pratt coefficient of absolute risk aversion for gambles over income or wealth in period t. In contrast, the expression $-u_{11}/u_1$ is the risk aversion coefficient for a hypothetical gamble in which the household is *forced to consume immediately* the outcome of the gamble. Clearly, it is the former concept that corresponds to the stochastic payoffs of a standard asset, such as a stock or bond, in a DSGE model. In order for $-u_{11}/u_1$ to be the relevant measure for pricing a security, it is not enough that the security pay off in units of consumption in period $t+1$. The household would additionally have to be prevented from adjusting its consumption and labor choices in period $t+1$ in response to the security's payoffs, so that the household is forced to absorb those payoffs into period $t+1$ consumption. It is difficult to imagine such a security—all standard securities in financial markets correspond to gambles over income or wealth, for which the $-V_{11}/V_1$ measure of risk aversion is the appropriate one.

4.2 Risk Aversion, the Stochastic Discount Factor, and Risk Premia

We now turn to the relationship between risk aversion, the labor margin, and asset pricing in the standard Lucas-Breeden stochastic discounting framework.

Let $m_{t+1} = \beta u_1(c^*_{t+1}, l^*_{t+1})/u_1(c^*_t, l^*_t)$ denote the household's stochastic discount factor and let p_t denote the cum-dividend price of a risky asset at time t, with $E_t p_{t+1}$ normalized to unity. The percentage difference between the risk-neutral price of the asset and its actual price—the risk premium on the asset—is given by:

$$\big(E_t m_{t+1} E_t p_{t+1} - E_t m_{t+1} p_{t+1}\big)/E_t m_{t+1} \;=\; -\mathrm{Cov}_t(dm_{t+1}, dp_{t+1})/E_t m_{t+1} \tag{37}$$

where Cov_t denotes the covariance conditional on information at time t, and $dx \equiv x_{t+1} - E_t x_{t+1}$, $x \in \{m, p\}$. For small changes dc^*_{t+1} and dl^*_{t+1}, we have, to first order:

$$dm_{t+1} = \frac{\beta}{u_1(c^*_t, l^*_t)} \left[u_{11}(c^*_{t+1}, l^*_{t+1}) dc^*_{t+1} + u_{12}(c^*_{t+1}, l^*_{t+1}) dl^*_{t+1} \right], \tag{38}$$

conditional on information at time t. In (38), the household's labor margin affects m_{t+1} and hence asset prices for two reasons: First, if $u_{12} \neq 0$, changes in l_{t+1} directly affect the household's marginal utility of consumption. Second, even if $u_{12} = 0$, the presence of the labor margin affects how the household responds to shocks and hence affects dc^*_{t+1}.

Intuitively, one can already see the relationship between risk aversion and dm_{t+1} in (38): if $dl^*_{t+1} = -\lambda dc^*_{t+1}$ and $dc^*_{t+1} = r da_{t+1}/(1 + w\lambda)$, as in Section 3, then dm_{t+1} equals the coefficient of absolute risk aversion, $\frac{u_{11} - \lambda u_{12}}{u_1} \frac{r}{1+w\lambda}$, times da_{t+1}. In actuality, the relationship is more complicated than this because θ (and hence w, r, and d) may change as well as a in response to macroeconomic shocks. For example, differentiating (12) and evaluating at steady state implies:

$$dl^*_{t+1} = -\lambda dc^*_{t+1} - \frac{u_1}{u_{22} + w u_{12}} dw_{t+1}, \tag{39}$$

to first order. The expression for dc^*_{t+1} is somewhat more complicated:

Lemma 6. *To first order, evaluated at the steady state,*

$$dc^*_{t+1} = \frac{r}{1+w\lambda}\left[da_{t+1} + E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1+r)^k}\left(l\,dw_{t+k} + dd_{t+k} + adr_{t+k}\right)\right] \tag{40}$$

$$+ \frac{u_1 u_{12}}{u_{11}u_{22} - u_{12}^2}\,dw_{t+1} \; + \; \frac{-u_1}{u_{11} - \lambda u_{12}}\,E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1+r)^k}\left(\frac{r\lambda}{1+w\lambda}\,dw_{t+k} - \beta dr_{t+k+1}\right),$$

PROOF: The expression follows from (2), (3), and (15). See the Appendix for details.

For the Arrow-Pratt one-shot gamble considered in Section 3, the aggregate variables w, r, and d were held constant, so (39)–(40) reduce to (13) and (21) in that case. The term in square brackets in (40) describes the change in the present value of household income, and thus the first line of (40) describes the income effect on consumption. The last line of (40) describes the substitution effect: changes in consumption due to changes in current and future wages and interest rates. (Recall $-u_1/\big(c(u_{11} - \lambda u_{12})\big)$ is the intertemporal elasticity of substitution.)

We are now in a position to relate risk aversion to asset prices and risk premia:

Proposition 7. *The household's stochastic discount factor satisfies*

$$dm_{t+1} \;=\; \beta\,\frac{u_{11} - \lambda u_{12}}{u_1}\,\frac{r}{1+w\lambda}\left[da_{t+1} + E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1+r)^k}\left(l\,dw_{t+k} + dd_{t+k} + adr_{t+k}\right)\right]$$

$$-\;\beta\,E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1+r)^k}\left(\frac{r\lambda}{1+w\lambda}\,dw_{t+k} - \beta dr_{t+k+1}\right) \tag{41}$$

to first order, evaluated at steady state. The risk premium in (37) *is given to second order around the steady state by:*

$$\frac{-u_{11} + \lambda u_{12}}{u_1}\,\frac{r}{1+w\lambda}\,\mathrm{Cov}_t(dp_{t+1}, d\hat{A}_{t+1}) \;+\; \mathrm{Cov}_t(dp_{t+1}, d\Psi_{t+1}), \tag{42}$$

where $d\hat{A}_{t+1}$ denotes the change in wealth given by the quantity in square brackets in (41) *and $d\Psi_{t+1}$ denotes the change in wages and interest rates given by the second line of* (41)*.*

PROOF: Substituting (39)–(40) into (38) yields (41). Substituting (41) into (37) yields (42). Note that $\beta = E_t m_{t+1}$. Finally, $\mathrm{Cov}(dx, dy)$ is accurate to second order when dx and dy are accurate to first order.

Proposition 7 shows the importance of risk aversion, as defined in Section 3, for asset prices. Risk premia increase linearly with the coefficient of absolute risk aversion—the first term in (42)—near the steady state.[14] This link should not be too surprising: Propositions 1–2 describe the risk premium for the simplest gambles over household wealth, while Proposition 7 shows that the same coefficient applies to more general gambles over financial assets that may be correlated with aggregate variables such as interest rates, wages, and net transfers.[15]

Proposition 7 also generalizes Merton's (1973) ICAPM to the case of variable labor. In (42), the first term is the coefficient of absolute risk aversion times the covariance of the asset price with household wealth, while the second term captures the asset's ability to hedge against intertemporal shocks (Merton's "changes in investment opportunities"). The first term can vanish if households are Arrow-Pratt risk neutral (that is, risk neutral in a cross-sectional or CAPM sense), but the second term remains nonzero because an asset that pays off well when future wages are low or interest rates are high (and hence future consumption is low) is preferable to an asset that pays off poorly in those situations.

Finally, Proposition 7 implies that it is no harder or easier to match asset prices in a dynamic equilibrium model with labor than it is in such a model without labor. A given level of risk aversion in a DSGE model with labor, measured correctly, will generate just as large a risk premium as the same level of risk aversion in a DSGE model without labor, for a given set of model covariances. Thus, the equity premium is not any harder to match, or any more puzzling, in dynamic models with labor than in consumption-only models, except to the extent that macroeconomic covariances may be harder to match.

We conclude this section by noting that the risk premium is essentially linear in relative as well as absolute risk aversion, using an appropriate measure of covariance:

Corollary 8. *In terms of relative risk aversion, the risk premium in* (42) *can be written as:*

$$\frac{-u_{11} + \lambda u_{12}}{u_1} \, \frac{c}{1 + w\lambda} \, \mathrm{Cov}_t\left(dp_{t+1}, \frac{d\hat{A}_{t+1}}{A}\right) \, + \, \mathrm{Cov}_t(dp_{t+1}, d\Psi_{t+1}), \tag{43}$$

or:

$$\frac{-u_{11} + \lambda u_{12}}{u_1} \, \frac{c + w(\bar{l} - l)}{1 + w\lambda} \, \mathrm{Cov}_t\left(dp_{t+1}, \frac{d\hat{A}_{t+1}}{\tilde{A}}\right) \, + \, \mathrm{Cov}_t(dp_{t+1}, d\Psi_{t+1}), \tag{44}$$

[14] This relationship also holds for the more general case of Epstein-Zin preferences, where it is easier to imagine varying risk aversion while holding the covariances in the model constant. See Section 5, below, and Rudebusch and Swanson (2009).

[15] Boldrin, Christiano, and Fisher (1997) argue that it is u_{11}/u_1 rather than V_{11}/V_1 that matters for the equity premium in their Figure 2. As shown here, it is V_{11}/V_1 that is crucial. What explains Boldrin et al.'s Figure 2 is that the covariance of equity prices with the short-term interest rate is not held constant—in particular, the variance of the risk-free rate changes greatly over the points in their Figure 2.

where A and $\tilde{A}$ are as in Definitions 2–3, and $d\hat{A}_{t+1}$ and $d\Psi_{t+1}$ are as defined in Proposition 7.

PROOF: See Appendix.[16]

5. Risk Aversion with Generalized Recursive Preferences

We now turn to the case of generalized recursive preferences, as in Epstein and Zin (1989) and Weil (1989). The household's asset accumulation equation (2) and no-Ponzi condition (3) are the same as in Section 2, but instead of maximizing (1), the household chooses c_t and l_t to maximize the generalized Bellman equation:

$$V(a_t;\theta_t) \;=\; \max_{(c_t,l_t)\in\Gamma(a_t;\theta_t)} \; u(c_t,l_t) + \beta\left(E_t\, V(a_{t+1};\theta_{t+1})^{1-\alpha}\right)^{1/(1-\alpha)}, \tag{45}$$

where $\alpha \in \mathbb{R}$, $\alpha \neq 1$.[17] Note that (45) is the same as (4), but with the value function "twisted" and "untwisted" by the coefficient $1-\alpha$. When $\alpha = 0$, the preferences given by (45) reduce to the special case of expected utility.

For the household's optimization problem to be well-defined, we require:

Assumption 7. *The generalized value function $V : X \to \mathbb{R}$ satisfying* (45) *exists.*

If $u \geq 0$ everywhere, then Epstein and Zin (1989) and Marinacci and Montruchhio (2010) provide sufficient conditions for Assumption 7 to be satisfied in a consumption-only framework (that is, with labor fixed). Sufficient conditions for the case of variable labor supply have not yet been derived in the literature.

If $u \leq 0$ everywhere, then it is natural to let $V \leq 0$ and replace (45) with:

$$V(a_t;\theta_t) \;=\; \max_{(c_t,l_t)\in\Gamma(a_t;\theta_t)} \; u(c_t,l_t) - \beta\left(E_t(-V(a_{t+1};\theta_{t+1}))^{1-\alpha}\right)^{1/(1-\alpha)}, \tag{46}$$

and sufficient conditions for Assumption 7 are similar.

[16] We write $d\hat{A}_{t+1}/A$ in (43) rather than $d\log A_{t+1}$ (and similarly in (44)) because $d\hat{A}_{t+1}$ differs slightly from dA_t. See the proof for details.

[17] We exclude the case $\alpha = 1$ here for simplicity. Note that, traditionally, Epstein-Zin preferences over consumption streams have been written as:

$$\widetilde{V}(a_t;\theta_t) = \max_{c_t}\left[c_t^{\rho} + \beta\left(E_t\widetilde{V}(a_{t+1};\theta_{t+1})^{\widetilde{\alpha}}\right)^{\rho/\widetilde{\alpha}}\right]^{1/\rho},$$

but by setting $V = \widetilde{V}^{\rho}$ and $\alpha = 1 - \widetilde{\alpha}/\rho$, this can be seen to correspond to (45).

To avoid the possibility of complex numbers arising in the maximand of (45) or (46), we require:

Assumption 8. *Either* $u : \Omega \to \mathbb{R}_+$, *or* $u : \Omega \to \mathbb{R}_-$.

The main advantage of generalized recursive preferences (45) is that they allow for greater flexibility in modeling risk aversion and the intertemporal elasticity of substitution. In (45), the intertemporal elasticity of substitution over deterministic consumption paths is exactly the same as in (4), but the household's risk aversion with respect to gambles can be amplified (or attenuated) by the additional parameter α.

5.1 Coefficients of Absolute and Relative Risk Aversion

Risk aversion continues to be given by Definition 1, where V is understood to mean the more general formulation in (45) or (46). The following proposition shows that risk aversion is well-defined and satisfies a generalized version of equations (8)–(9):

Proposition 9. *Let* $(a_t; \theta_t)$ *be an interior point of* X *and let* V *be given by* (45) *or* (46)*. The household's coefficient of absolute risk aversion at* $(a_t; \theta_t)$ *exists and equals:*

$$\frac{-E_t V(a^*_{t+1}; \theta_{t+1})^{-\alpha} \left[V_{11}(a^*_{t+1}; \theta_{t+1}) \; - \; \alpha \, \frac{V_1(a^*_{t+1}; \theta_{t+1})^2}{V(a^*_{t+1}; \theta_{t+1})} \right]}{E_t V(a^*_{t+1}; \theta_{t+1})^{-\alpha} V_1(a^*_{t+1}; \theta_{t+1})} \,. \tag{47}$$

Evaluated at steady state, (47) *simplifies to:*

$$\frac{-V_{11}(a;\theta)}{V_1(a;\theta)} \; + \; \alpha \, \frac{V_1(a;\theta)}{V(a;\theta)} \,. \tag{48}$$

Proof: See Appendix.

The first term in (48) is the same as the expected utility case (9), while the second term in (48) reflects the amplification or attenuation of risk aversion from the additional curvature parameter α. When $\alpha = 0$, (47)–(48) reduce to (8)–(9). When $u \geq 0$ and hence $V \geq 0$, higher values of α correspond to greater degrees of risk aversion; when u and $V \leq 0$, the opposite is true: higher values of α correspond to lesser degrees of risk aversion.

Proposition 9 is particularly important because, unlike Proposition 1, there is no pre-existing "folk wisdom" in the profession regarding how to compute risk aversion for Epstein-Zin preferences with labor. Risk aversion for these preferences has only been computed previously in homothetic, isoelastic, consumption-only models where the value function can be computed in closed form. Proposition 9 and Proposition 10, below, do not require u to be homothetic or the rest of the model to be homogenous, are valid for general and unknown functional forms V, and allow for the presence of labor.

Equation (48) also highlights an important feature of risk aversion with generalized recursive preferences: it is not invariant with respect to additive shifts of the utility kernel, except for the special case of expected utility ($\alpha = 0$). because the level of V enters into the right-hand side of (48). That is, the utility kernels $u(\cdot,\cdot)$ and $u(\cdot,\cdot)+k$, where k is a constant, lead to different household attitudes toward risk. The household's preferences are invariant, however, with respect to multiplicative transformations of the utility kernel.

We now turn to computing closed-form expressions for (48). Straightforward calcul-cations along the lines of Section 3 show that expressions (10)–(14) and (16)–(21) for V_1, V_{11}, $\partial l_t^*/\partial a_t$, and $\partial c_t^*/\partial a_t$ remain valid, even though the Euler equation itself is slightly different.[18] Moreover, $V = u(c,l)/(1-\beta)$ at the steady state. Substituting these into (48) establishes:

Proposition 10. *The household's coefficient of absolute risk aversion in Proposition 9, evaluated at steady state, is given by:*

$$\frac{-V_{11}}{V_1} + \alpha\frac{V_1}{V} \;=\; \frac{-u_{11}+\lambda u_{12}}{u_1}\,\frac{r}{1+w\lambda} \;+\; \alpha\,\frac{r\,u_1}{u}\,. \tag{49}$$

Relative risk aversion likewise continues to be given by Definitions 2–3, where V is understood to mean the more general formulation in (45) or (46), and where wealth is defined using the stochastic discount factor corresponding to Epstein-Zin preferences.[19]

Corollary 11. *The household's consumption-based coefficient of relative risk aversion, evaluated at steady state, is given by:*

$$\frac{-AV_{11}}{V_1} + \alpha\frac{AV_1}{V} \;=\; \frac{-u_{11}+\lambda u_{12}}{u_1}\,\frac{c}{1+w\lambda} \;+\; \alpha\,\frac{c\,u_1}{u}\,. \tag{50}$$

The household's leisure-and-consumption-based coefficient of relative risk aversion, evaluated at steady state, is given by $(c+w(\bar{l}-l))/c$ *times* (50).

[18] The household's Euler equation is given by:

$$u_1(c_t^*,l_t^*) = \beta E_t(1+r_{t+1})[V(a_{t+1}^*;\theta_{t+1})/(E_tV(a_{t+1}^*;\theta_{t+1})^{1-\alpha})^{1/(1-\alpha)}]^{-\alpha}u_1(c_{t+1}^*,l_{t+1}^*).$$

[19] The household's stochastic discount factor is given by:

$$m_{t,t+1} = \beta u_1(c_{t+1}^*,l_{t+1}^*)[V(a_{t+1}^*;\theta_{t+1})/(E_tV(a_{t+1}^*;\theta_{t+1})^{1-\alpha})^{1/(1-\alpha)}]^{-\alpha}/u_1(c_t^*,l_t^*).$$

At steady state, however, this simplifies to the usual β.

5.2 Examples

Example 3. Consider the additively separable utility kernel:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}, \tag{51}$$

with generalized recursive preferences (46) and $\chi > 0$, $\eta > 0$, and $\gamma > 1$, which was used by Rudebusch and Swanson (2009).[20] In this case, $u(\cdot,\cdot) < 0$, risk aversion is decreasing in α, and $\alpha < 0$ corresponds to preferences that are more risk averse than expected utility.

In models without labor, period utility $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma)$ implies a coefficient of relative risk aversion of $\gamma + \alpha(1-\gamma)$, which we will refer to as the traditional measure.[21] Taking into account both the consumption and labor margins of (51), the household's consumption-based coefficient of relative risk aversion (50) is given by:

$$\begin{aligned} \frac{-AV_{11}}{V_1} + \alpha \frac{AV_1}{V} &= \frac{\gamma}{1 + \frac{\gamma}{\chi}\frac{wl}{c}} + \frac{\alpha(1-\gamma)}{1 + \frac{\gamma-1}{1+\chi}\frac{wl}{c}}, \\ &\approx \frac{\gamma}{1 + \frac{\gamma}{\chi}} + \frac{\alpha(1-\gamma)}{1 + \frac{\gamma-1}{1+\chi}}, \end{aligned} \tag{52}$$

using $c \approx wl$. As in Example 1, the household's leisure-and-consumption-based coefficient of relative risk aversion is not well defined in this example, so we restrict attention to the consumption-based measure (52).

As χ becomes large, household labor becomes less flexible and the bias from ignoring the labor margin shrinks to zero. As χ approaches zero, (52) decreases to $\alpha(1-\gamma)/\gamma$, which is close to zero if we think of γ as being close to unity. Thus, for given values of γ and α, actual risk aversion can lie anywhere between about zero and $\gamma + \alpha(1-\gamma)$, depending on χ.

Example 4. Van Binsbergen et al. (2010) and Backus, Routledge, and Zin (2008) consider generalized recursive preferences with:

$$u(c_t, l_t) = \frac{\left(c_t^{\nu}(1-l_t)^{1-\nu}\right)^{1-\gamma}}{1-\gamma}, \tag{53}$$

[20] We restrict attention here to the case $\gamma > 1$, consistent with Assumption 8. The case $\gamma \leq 1$ can be considered if we place restrictions on the domain of c_t and l_t such that $u(\cdot,\cdot) < 0$; one can always choose units for c_t and l_t such that this doesn't represent much of a constraint in practice. Of course, one can also consider alternative utility kernels with $\gamma \leq 1$ for which $u(\cdot,\cdot) > 0$.

[21] Set $\eta = 0$ and $\lambda = 0$ and substitute (51) into (50). This is the case, for example, in Epstein and Zin (1989) and Boldrin, Christiano, and Fisher (1997), which do not have labor. In models with variable labor, Rudebusch and Swanson (2009) refer to $\gamma + \alpha(1-\gamma)$ as the *quasi* coefficient of relative risk aversion.

where $\gamma > 0$, $\gamma \neq 1$, and $\nu \in (0,1)$. Van Binsbergen et al. call $\gamma + \alpha(1-\gamma)$ the coefficient of relative risk aversion, while Backus et al. use $\gamma\nu + \alpha(1-\gamma)\nu + (1-\nu)$, after mapping each study's notation over to the present paper's. The former measure effectively treats consumption and leisure as a single composite commodity, while the latter measure allows ν to affect the household's attitudes toward risk.

Substituting (53) into (50), the household's consumption-based coefficient of relative risk aversion is:

$$\frac{-AV_{11}}{V_1} + \alpha \frac{AV_1}{V} \;=\; \gamma\nu + \alpha(1-\gamma)\nu, \tag{54}$$

while the leisure-and-consumption-based coefficient of relative risk aversion is:[22]

$$\frac{-\tilde{A}V_{11}}{V_1} + \alpha \frac{\tilde{A}V_1}{V} \;=\; \gamma + \alpha(1-\gamma). \tag{55}$$

The latter agrees with the Van Binsbergen et al. (2010) measure of risk aversion, while the former is similar to (though not quite the same as) the Backus et al. (2008) measure. In this paper, we have provided the formal justification for both measures, (54) and (55). Note that the leisure-and-consumption-based measure of risk aversion again corresponds to treating the Cobb-Douglas aggregate of consumption and leisure as a single, composite good.

Example 5. Tallarini (2000) considers an alternative Epstein-Zin specification:

$$\tilde{V}(a_t;\theta_t) \;\equiv\; u(c_t^*, l_t^*) \;+\; \frac{\beta(1+\theta)}{(1-\beta)(1-\chi)} \log E_t \exp\left[\frac{(1-\beta)(1-\chi)}{1+\theta}\,\tilde{V}(a_{t+1}^*;\theta_{t+1})\right], \tag{56}$$

with utility kernel:

$$u(c_t, l_t) \;=\; \log c_t + \theta \log(\bar{l} - l_t). \tag{57}$$

We can compute the coefficient of absolute risk aversion for (56) by following along the steps in the proof of Proposition 9, which yields:

$$\frac{-\tilde{V}_{11}(a;\theta)}{\tilde{V}_1(a;\theta)} \;-\; \frac{(1-\beta)(1-\chi)}{1+\theta}\,\tilde{V}_1(a;\theta). \tag{58}$$

The other steps leading up to Proposition 10 are the same, so substituting in for $\tilde{V}_1$ and $\tilde{V}_{11}$ in (58) yields a consumption-based coefficient of relative risk aversion of:

$$\frac{-u_{11} + \lambda u_{12}}{u_1}\,\frac{c}{1+w\lambda} \;-\; \frac{1-\chi}{1+\theta}\,cu_1 \;=\; \frac{\chi}{1+\theta}\,. \tag{59}$$

[22] As $\nu \to 0$, $w/c \to \infty$, so consumption becomes trivial to insure with variations in labor supply. This explains why the consumption-based coefficient of relative risk aversion in (53) vanishes as $\nu \to 0$.

The leisure-and-consumption-based coefficient of relative risk aversion is χ, which again corresponds to treating consumption and leisure as a single, composite good.

Both coefficients of relative risk aversion differ from the value $(\chi+\theta)/(1+\theta)$ emphasized by Tallarini (2000). Tallarini applies the traditional, one-good measure of risk aversion for Epstein-Zin preferences, $\frac{-cu_{11}}{u_1} - \frac{1-\chi}{1+\theta}\, cu_1$, to the case where $\theta > 0$ but labor is fixed. This ignores the fact that, when $\theta > 0$, households will vary their labor endogenously in response to shocks.

6. Risk Aversion with Habits

Many studies in macroeconomics and finance assume that households derive utility not from consumption itself, but from consumption relative to some reference level, or habit stock. Habits, in turn, can have substantial effects on the household's attitudes toward risk (e.g., Campbell and Cochrane, 1999, Boldrin, Christiano, and Fisher, 1997). In this section, we investigate how habits affect risk aversion in the DSGE framework.

We generalize the household's utility kernel in this section to $u(c_t - h_t, l_t)$, where h_t denotes the household's reference level of consumption, or habits. We focus on an additive rather than multiplicative specification for habits because the implications for risk aversion are typically more interesting in the additive case.

If the habit stock h_t is external to the household ("keeping up with the Joneses" utility), then the parameters that govern the process for h_t can be incorporated into the exogenous state vector θ_t, and the analysis proceeds essentially as in the previous sections. However, if the habit stock h_t is a function of the household's own past levels of consumption, then the state variables of the household's optimization problem must be augmented to include the state variables that govern h_t. We consider each of these cases in turn.

6.1 External Habits

When the reference consumption level h_t in utility $u(c_t - h_t, l_t)$ is external to the household, then the parameters that govern h_t can be incorporated into the exogenous state vector θ_t and the analysis of the previous sections carries over essentially as before. In particular, the coefficient of absolute risk aversion continues to be given by Proposition 1 in the case of expected utility and Proposition 9 for generalized recursive preferences. The household's intratemporal optimality condition (12) still implies (13)–(14), the household's Euler equation (15) still implies (16)–(19), and the household's budget constraint (2)–(3) thus implies (21), just as in Section 3.

The only real differences that arise relative to the case without habits is, first, that the steady-state point at which the derivatives of $u(\cdot,\cdot)$ are evaluated is $(c-h, l)$ rather than (c, l), and second, that relative risk aversion confronts the household with a hypothetical gamble over c rather than $c-h$, which has a tendency to make the household more risk averse for a given functional form $u(\cdot,\cdot)$, because the stakes are effectively larger.

Example 6. Consider the case of expected utility with additively separable utility kernel:

$$u(c_t - h_t, l_t) \;=\; \frac{(c_t - h_t)^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}\,, \tag{60}$$

where γ, χ, $\eta > 0$. The traditional measure of risk aversion for this example is $-cu_{11}/u_1 = \gamma c/(c-h)$, which exceeds γ by a factor that depends on the importance of habits relative to consumption. The consumption-based coefficient of relative risk aversion is:

$$\begin{aligned} \frac{-AV_{11}}{V_1} &= \frac{-cu_{11}}{u_1} \frac{1}{1 + w\,\frac{wu_{11}}{u_{22}}}\,, \\ &= \frac{\gamma c}{(c-h)} \frac{1}{1 + \frac{\gamma c}{\chi(c-h)} \frac{wl}{c}}\,. \end{aligned} \tag{61}$$

When there is a labor margin in the model, the household's consumption-based coefficient of relative risk aversion (61) is less than the traditional measure by the factor $1 + \frac{\gamma c}{\chi(c-h)}$, using $wl \approx c$. Ignoring the labor margin in (61) thus leads to an even greater bias in the model with habits ($h > 0$) than without habits ($h = 0$). If $\gamma = 2$, $\chi = 1$, and $h = .8c$, then the household's true risk aversion is less than the traditional measure by a factor of 11.

With generalized recursive preferences rather than expected utility preferences, the consumption-based coefficient of relative risk aversion for (60) is:

$$\frac{\gamma c}{(c-h)} \frac{1}{1 + \frac{\gamma c}{\chi(c-h)} \frac{wl}{c}} + \frac{\alpha(1-\gamma)c}{(c-h)} \frac{1}{1 + \frac{c}{(c-h)} \frac{\gamma-1}{1+\chi} \frac{wl}{c}}\,. \tag{62}$$

Again, the bias from ignoring the labor margin in (62) is even greater in the model with habits ($h > 0$) than without habits ($h = 0$).

6.2 Internal Habits

When habits are internal to the household, we must specify how the household's actions affect its future habits. We assume that the habit stock evolves according to the standard autoregressive process:

$$h_t = \rho h_{t-1} + b c_{t-1}, \tag{63}$$

where $\rho \in (-1, 1)$, $b \in (0, 1)$, and $\rho + b < 1$ to ensure $h < c$ in steady state.

With internal habits, the value of h_{t+1} depends on the household's choices in period t, so we write out the dependence of the household's value function on h_t explicitly:

$$V(a_t, h_t; \theta_t) = u(c_t^* - h_t, l_t^*) + \beta \left(E_t V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^{1-\alpha} \right)^{1/(1-\alpha)}, \tag{64}$$

where $c_t^* \equiv c^*(a_t, h_t; \theta_t)$ and $l_t^* \equiv l^*(a_t, h_t; \theta_t)$ denote the household's optimal choices for consumption and labor in period t as functions of the household's state vector, and a_{t+1}^* and h_{t+1}^* denote the optimal stocks of assets and habits in period $t+1$ that are implied by c_t^* and l_t^*; that is, $a_{t+1}^* \equiv (1 + r_t)a_t + w_t l_t^* + d_t - c_t^*$ and $h_{t+1}^* \equiv \rho h_t^* + b c_t^*$. Assumptions 2, 3, 4, and 7 must be modified slightly to include the additional state variable h_t, but these modifications are straightforward.

We apply Definition 1 again and solve for the household's coefficient of absolute risk aversion in exactly the same manner as Propositions 1 and 9:

Proposition 12. *Let $(a_t; h_t; \theta_t)$ be an interior point of X and let V be given by (45) or (46). The household's coefficient of absolute risk aversion at $(a_t; h_t; \theta_t)$ exists and equals:*

$$\frac{-E_t V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^{-\alpha} \left[V_{11}(a_{t+1}^*, h_{t+1}^*; \theta_{t+1}) \; - \; \alpha \, \dfrac{V_1(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^2}{V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})} \right]}{E_t V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^{-\alpha} V_1(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})}. \tag{65}$$

Evaluated at steady state, (65) simplifies to:

$$\frac{-V_{11}(a, h; \theta)}{V_1(a, h; \theta)} \; + \; \alpha \, \frac{V_1(a, h; \theta)}{V(a, h; \theta)}. \tag{66}$$

Proof: Essentially identical to the proof of Proposition 9.

Computing closed-form expressions for V_1 and V_{11} in (66) follows the same general methodology as in Section 3, but is more complicated in the presence of internal habits because of the dynamic relationship between the household's current consumption and its future habits.

Proposition 13. *The household's coefficient of absolute risk aversion in Proposition 12, evaluated at steady state, is given by:*

$$\frac{-V_{11}}{V_1} + \alpha\frac{V_1}{V} \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1}\;\frac{\left(1 - \frac{\beta b}{1-\beta\rho}\right) r}{1 + \left(1 - \frac{\beta b}{1-\beta\rho}\right) w\lambda} \;+\; \alpha\frac{r\,u_1}{u}\left(1 - \frac{\beta b}{1-\beta\rho}\right). \tag{67}$$

Proof: See Appendix.

Corollary 14. *The household's consumption-based coefficient of relative risk aversion, evaluated at steady state, is given by:*

$$\frac{-AV_{11}}{V_1} + \alpha\frac{AV_1}{V} \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1}\;\frac{\left(1 - \frac{\beta b}{1-\beta\rho}\right) c}{1 + \left(1 - \frac{\beta b}{1-\beta\rho}\right) w\lambda} \;+\; \alpha\frac{c\,u_1}{u}\left(1 - \frac{\beta b}{1-\beta\rho}\right). \tag{68}$$

The household's leisure-and-consumption-based coefficient of relative risk aversion, evaluated at steady state, is given by $(c + w(\bar{l} - l))/c$ *times* (68).

Equations (67)–(68) have essentially the same form as the corresponding expressions in the model without habits.

Example 7. Consider the utility kernel of example 4.1:

$$u(c_t - h_t, l_t) \;=\; \frac{(c_t - h_t)^{1-\gamma}}{1-\gamma} \;-\; \eta\frac{l_t^{1+\chi}}{1+\chi}\,, \tag{69}$$

where γ, χ, $\eta > 0$, but now with habits $h_t = bc_{t-1}$ internal to the household rather than external. In thise case, the household's consumption-based coefficient of relative risk aversion is given by:

$$\begin{aligned}
\frac{-AV_{11}}{V_1} &= \frac{-cu_{11}}{u_1}\;\frac{1-\beta b}{1 + (1-\beta b)w\lambda}\,, \\
&= \gamma\,\frac{1-\beta b}{1-b}\;\frac{1}{1 + \frac{\gamma}{\chi}\frac{1-\beta b}{1-b}\frac{wl}{c}}\,, \\
&\approx \frac{\gamma}{1 + \frac{\gamma}{\chi}}\,,
\end{aligned} \tag{70}$$

where the last line uses $\beta \approx 1$ and $wl \approx c$.

The most striking feature of equation (70) is that it is independent of b, the importance of habits. This is in sharp contrast to the case of external habits, where risk aversion is strongly increasing in b (cf. equation (61)).

7. Risk Aversion Away from the Steady State

The simple, closed-form expressions for risk aversion derived above hold exactly only at the model's nonstochastic steady state. For values of $(a_t; \theta_t)$ away from steady state, these expressions are only approximations. In this section, we evaluate the accuracy of those approximations by computing risk aversion numerically for a standard real business cycle model.

There is a unit continuum of representative households in the model, each with optimization problem (1)–(3), with additively separable utility kernel (30) from Example 1. The economy contains a unit continuum of perfectly competitive firms, each with production function $y_t = A_t k_t^{1-\phi} l_t^{\phi}$, where y_t, l_t, and k_t denote firm output, labor, and beginning-of-period capital, and A_t denotes an exogenous technology process that follows $\log A_t = \rho \log A_{t-1} + \varepsilon_t$, where ε_t is i.i.d. with mean zero and variance σ_ε^2. Labor and capital are supplied by households at the competitive wage and rental rates w_t and r_t^k. Capital is the only asset, which households accumulate according to $k_{t+1} = (1 + r_t)k_t + w_t l_t - c_t$, where $r_t = r_t^k - \delta$, δ is the capital depreciation rate, and c_t denotes household consumption.

We set $\beta = .99$, $\gamma = 2$, and $\chi = 1.5$, corresponding to an intertemporal elasticity of substitution of 0.5 and Frisch elasticity of 2/3. We set $\eta = .4514$ to normalize steady-state labor $l = 1$. We set $\phi = .7$, $\delta = .025$, $\rho = .9$, and $\sigma_\varepsilon = .01$.

The household's consumption-based coefficient of relative risk aversion, evaluated at steady state, is given by (31). For the parameter values above, this implies a risk aversion coefficient of .9145, less than half the traditional measure of $\gamma = 2$. Away from the steady state, equations (8) and (10)–(17) remain valid, and we use them to compute the household's coefficient of relative risk aversion by solving for V_1, V_{11}, λ_t, and $\partial c_t^* / \partial a_t$ numerically (see the Appendix for details). Figure 2 graphs the result as a function of $\log(k_t/k)$ and $\log A_t$ over a wide range of values for these state variables, ± 50 percentage points in logarithmic terms (equal to about 15 and 20 standard deviations of $\log k_t$ and $\log A_t$, respectively).[23] The horizontal dashed black lines in Figure 2 report the constant, closed-form value for risk aversion at the nonstochastic steady state, equal to .9145. The solid red lines in the figure denote the numerical solution for risk aversion for general values of k_t and A_t. The key

[23] The unconditional standard deviations of $\log A_t$ and $\log(k_t/k)$ are about 2.3 and 3.5 percent, respectively. The ergodic mean of $\log A_t$ is zero and that of $\log(k_t/k)$ is about .01, or 1 percent.

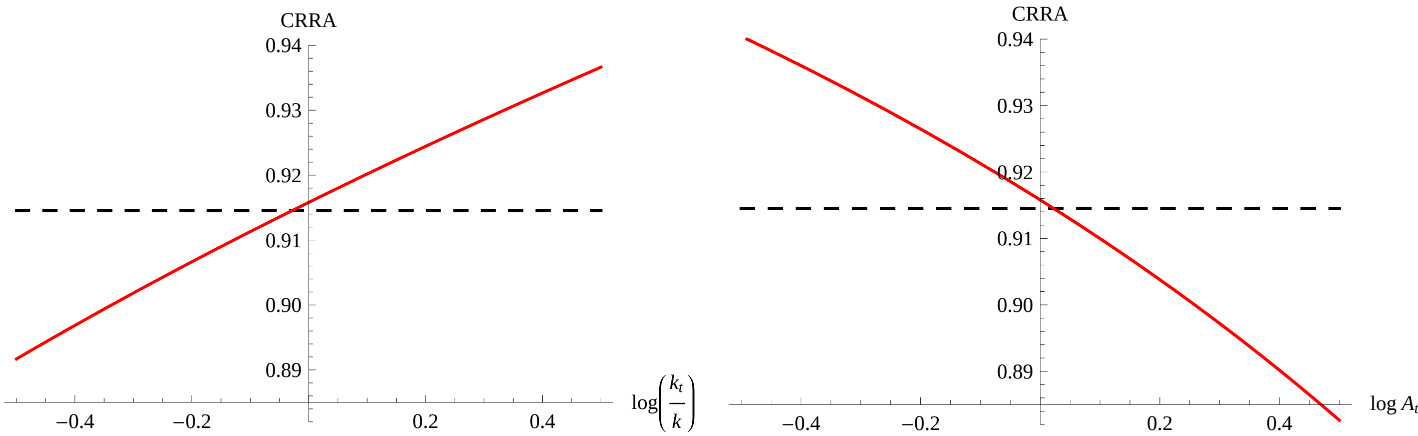

Figure 2. Consumption-based coefficient of relative risk aversion for the utility kernel $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$ as a function of k_t and A_t in a standard real business cycle model. Each graph holds the other state variable fixed at its steady-state value. Dashed black lines denote the constant, closed-form value for risk aversion evaluated at the nonstochastic steady state. Solid red lines denote the numerical solution for risk aversion for general k_t and A_t. See text for details.

observation is that, even over the very wide range of values for the state variables considered, the household's coefficient of relative risk aversion ranges between .88 and .94, very close to the steady-state value of .9145, and never near the traditional value of 2.[24] Thus, the closed-form expressions in Section 3 seem to provide a good approximation to the true level of household risk aversion in a standard model even far away from steady state.

In Figure 3, we extend this analysis to the case of Epstein-Zin preferences with a much higher level of risk aversion. The specification of the model and parameter values are the same as above, but with generalized recursive preferences (46) instead of expected utility (4). We set the Epstein-Zin curvature paramater $\alpha = -50$, which implies a traditional measure of risk aversion of 52, but actual consumption-based relative risk aversion of about 37.8 (see Example 3). Even for the very high level of risk aversion in this example, the closed-form expressions remain good approximations far away from steady state—the range of values plotted in Figure 3 corresponds to about ±5 standard deviations of the state variables, and the coefficient of relative risk aversion ranges between about 37.6 and 38.2 over this range, very close to the steady-state value and never near the traditional measure of 52.[25]

[24] The red lines do not intersect the black lines at the vertical axis because c_t^* and l_t^* evaluated at $k_t = k$ and $A_t = A$ do not equal the nonstochastic steady state values c and l due to the presence of uncertainty (e.g., precautionary savings).

[25] The unconditional standard deviations of $\log A_t$ and $\log(k_t/k)$ remain about 2.3 and 3.5 percent, respectively, and the ergodic means are 0 and about 1 percent, as before. We plot a narrower range of values for the state variables in Figure 3 because the much greater curvature of the model in this example reduces the

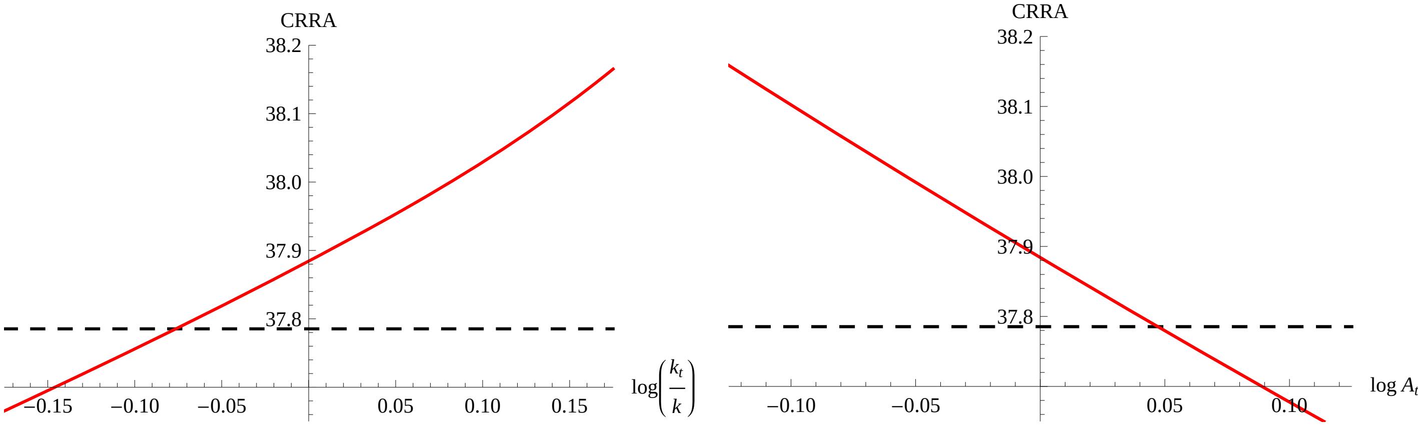

Figure 3. Consumption-based coefficient of relative risk aversion for Epstein-Zin preferences with high risk aversion and utility kernel $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$, as a function of k_t and A_t. See notes to Figure 2 and text for details.

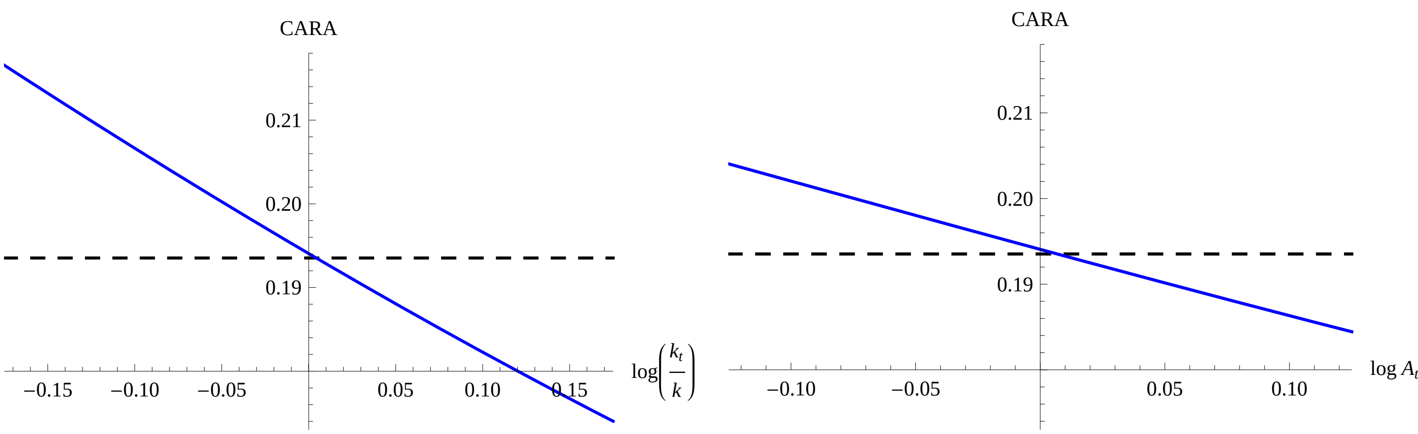

Figure 4. Coefficient of absolute risk aversion for Epstein-Zin preferences with high risk aversion and utility kernel $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$, as a function of k_t and A_t. See notes to Figure 2 and text for details.

Finally, it is worth noting that absolute risk aversion in these examples is countercyclical with respect to both k_t and A_t. Figure 4 graphs absolute risk aversion for the Epstein-Zin example above. Since risk aversion is so high in this example, the household is willing to pay about 19.4 cents to avoid a fair gamble with a standard deviation of one dollar. This willingness to pay varies from about 18 to 22 cents over the range of values for the state variables in Figure 4, with higher values of the states corresponding to higher household wealth and lower risk aversion. For the case of expected utility preferences (not shown), the coefficient of absolute risk aversion is much smaller, less than one-half of one cent to avoid the same gamble, but the pattern of variation is very similar to Figure 4, albeit smaller in

accuracy of our numerical solution method outside this range.

magnitude.

Looking back at Figures 2 and 3, relative risk aversion is not countercyclical in those figures with respect to k_t because household wealth is increasing in k_t and A_t. Indeed, for higher k_t, the increase in wealth is sufficiently large that the household's relative risk aversion increases with k_t, even though absolute risk aversion is decreasing.

8. Discussion and Conclusions

The traditional measure of risk aversion, $-cu_{11}/u_1$, ignores the household's ability to partially offset shocks to asset returns with changes in hours worked. For reasonable parameterizations, the traditional measure can overstate risk aversion by a factor of three or more. Many studies in the macroeconomics, macro-finance, and international literatures thus may overstate the actual degree of risk aversion in their models by a substantial degree.

Risk aversion matters for asset pricing. Asset prices in dynamic models can behave very differently depending on how the household's labor margin is specified. If risk aversion is measured incorrectly because the labor margin is ignored, then risk premia in the model are more likely to be puzzling. An extreme example of this is when household utility has a zero discriminant—implying risk neutrality—even when the traditional measure of risk aversion is large.

Risk neutrality itself can be a desirable feature for some applications, such as labor market search or financial frictions, because risk neutrality allows for simple or closed-form solutions to key aspects of the model. The present paper suggests new ways to model risk neutrality that do not require linearity of utility in consumption, which has undesirable implications for interest rates and consumption growth. Instead, any utility kernel with a singular Hessian can be used.

It is also worth noting two implications that do not follow from the analysis in the present paper. First, the paper does not find that it is any harder or easier to match risk premia in dynamic equilibrium models with labor than it is in models without labor (recall Proposition 7). Second, the paper does not shed any light on what plausible empirical values for risk aversion might be. Empirical estimates of risk aversion based on surveys, changes in income or wealth, or cash prizes are generally just as valid in the framework of this paper as they are in dynamic models without labor.

Finally, many of the observations of the present paper apply not just to dynamic models with labor, but to any such model with multiple goods in the utility function. Models with home production, money in the utility function, or tradeable and nontradeable goods can all imply very different household attitudes toward risk than traditional measures of risk aversion would suggest. The simple, closed-form expressions for risk aversion derived in this paper, and the methods of the paper more generally, are potentially useful in any of these contexts, in pricing any asset—stocks, bonds, or futures, in foreign or domestic currency—within the framework of dynamic equilibrium models. Since these models are a mainstay of research in academia, at central banks, and international financial institutions, the applicability of the results should be widespread.

Appendix: Proofs of Propositions and Numerical Solution Details

Proof of Proposition 1

Since $(a_t; \theta_t)$ is an interior point of X, $V(a_t + \frac{\sigma\underline{\varepsilon}}{1+r_t}; \theta_t)$ and $V(a_t + \frac{\sigma\overline{\varepsilon}}{1+r_t}; \theta_t)$ exist for sufficiently small σ, and $V(a_t + \frac{\sigma\underline{\varepsilon}}{1+r_t}; \theta_t) \leq \widetilde{V}(a_t; \theta_t; \sigma) \leq V(a_t + \frac{\sigma\overline{\varepsilon}}{1+r_t}; \theta_t)$, hence $\widetilde{V}(a_t; \theta_t; \sigma)$ exists. Moreover, since $V(\cdot;\cdot)$ is continuous and increasing in its first argument, the intermediate value theorem implies there exists a unique $-\mu(\sigma) \in [\sigma\underline{\varepsilon}, \sigma\overline{\varepsilon}]$ with $V(a_t - \frac{\mu(\sigma)}{1+r_t}; \theta_t) = \widetilde{V}(a_t; \theta_t; \sigma)$.

For a sufficiently small fee μ in (7), the first-order change in household welfare (5) is given by:

$$\frac{-V_1(a_t;\theta_t)}{1+r_t}\, d\mu\,. \tag{A1}$$

Using the envelope theroem, we can rewrite (A1) as:

$$-\beta E_t V_1(a^*_{t+1}; \theta_{t+1})\, d\mu\,. \tag{A2}$$

Turning now to the gamble in (6), note that the household's optimal choices for consumption and labor in period t, c^*_t and l^*_t, will generally depend on the size of the gamble σ—for example, the household may undertake precautionary saving when faced with this gamble. Thus, in this section we write $c^*_t \equiv c^*(a_t;\theta_t;\sigma)$ and $l^*_t \equiv l^*(a_t;\theta_t;\sigma)$ to emphasize this dependence on σ. The household's value function, inclusive of the one-shot gamble in (6), satisfies:

$$\widetilde{V}(a_t;\theta_t;\sigma) = u(c^*_t, l^*_t) + \beta E_t V(a^*_{t+1};\theta_{t+1}), \tag{A3}$$

where $a^*_{t+1} \equiv (1+r_t)a_t + w_t l^*_t + d_t - c^*_t$. Because (6) describes a one-shot gamble in period t, it affects assets a^*_{t+1} in period $t+1$ but otherwise does not affect the household's optimization problem from period $t+1$ onward; as a result, the household's value-to-go at time $t+1$ is just $V(a^*_{t+1};\theta_{t+1})$, which does not depend on σ except through a^*_{t+1}.

Differentiating (A3) with respect to σ, the first-order effect of the gamble on household welfare is:

$$\left[u_1 \frac{\partial c^*}{\partial\sigma} + u_2 \frac{\partial l^*}{\partial\sigma} + \beta E_t V_1 \cdot \left(w_t \frac{\partial l^*}{\partial\sigma} - \frac{\partial c^*}{\partial\sigma} + \varepsilon_{t+1}\right)\right] d\sigma, \tag{A4}$$

where the arguments of u_1, u_2, and V_1 are suppressed to reduce notation. Optimality of c^*_t and l^*_t implies that the terms involving $\partial c^*/\partial\sigma$ and $\partial l^*/\partial\sigma$ in (A4) cancel, as in the usual envelope theorem (these derivatives vanish at $\sigma = 0$ anyway, for the reasons discussed below). Moreover, $E_t V_1(a^*_{t+1};\theta_{t+1})\varepsilon_{t+1} = 0$ because ε_{t+1} is independent of θ_{t+1} and a^*_{t+1}, evaluating the latter at $\sigma = 0$. Thus, the first-order cost of the gamble is zero, as in Arrow (1964) and Pratt (1965).

To second order, the effect of the gamble on household welfare is:

$$\Bigg[u_{11}\left(\frac{\partial c^*}{\partial\sigma}\right)^2 + 2u_{12}\frac{\partial c^*}{\partial\sigma}\frac{\partial l^*}{\partial\sigma} + u_{22}\left(\frac{\partial l^*}{\partial\sigma}\right)^2 + u_1\frac{\partial^2 c^*}{\partial\sigma^2} + u_2\frac{\partial^2 l^*}{\partial\sigma^2}$$
$$+ \beta E_t V_{11}\cdot\left(w_t\frac{\partial l^*}{\partial\sigma} - \frac{\partial c^*}{\partial\sigma} + \varepsilon_{t+1}\right)^2 + \beta E_t V_1 \cdot \left(w_t \frac{\partial^2 l^*}{\partial\sigma^2} - \frac{\partial^2 c^*}{\partial\sigma^2}\right)\Bigg]\frac{d\sigma^2}{2}\,. \tag{A5}$$

The terms involving $\partial^2 c^*/\partial\sigma^2$ and $\partial^2 l^*/\partial\sigma^2$ cancel due to the optimality of c^*_t and l^*_t. The derivatives $\partial c^*/\partial\sigma$ and $\partial l^*/\partial\sigma$ vanish at $\sigma = 0$ (there are two ways to see this: first, the linearized version of the model is certainty equivalent; alternatively, the gamble in (6) is isomorphic for positive and negative σ, hence c^* and l^* must be symmetric about $\sigma = 0$, implying the derivatives vanish). Thus, for infinitesimal gambles, (A5) simplifies to:

$$\beta E_t V_{11}(a^*_{t+1};\theta_{t+1})\,\varepsilon^2_{t+1}\frac{d\sigma^2}{2}\,. \tag{A6}$$

Finally, ε_{t+1} is independent of θ_{t+1} and a^*_{t+1}, evaluating the latter at $\sigma = 0$. Since ε_{t+1} has unit variance, (A6) reduces to:

$$\beta E_t V_{11}(a^*_{t+1};\theta_{t+1})\frac{d\sigma^2}{2}\,. \tag{A7}$$

Equating (A2) to (A7) allows us to solve for $d\mu$ as a function of $d\sigma^2$. Thus, the limit $\lim_{\sigma\to 0} 2\mu(\sigma)/\sigma^2$ exists and is given by:

$$\frac{-\,E_t V_{11}(a^*_{t+1};\theta_{t+1})}{E_t V_1(a^*_{t+1};\theta_{t+1})}\,. \tag{A8}$$

To evaluate (A8) at the nonstochastic steady state, set $a_{t+1} = a$ and $\theta_{t+1} = \theta$ to get:

$$\frac{-\,V_{11}(a;\theta)}{V_1(a;\theta)}\,. \tag{A9}$$

Proof of Lemma 6

Differentiating the household's Euler equation (15) and evaluating at steady state yields:

$$u_{11}(dc^*_t - E_t dc^*_{t+1}) + u_{12}(dl^*_t - E_t dl^*_{t+1}) = \beta E_t u_1 dr_{t+1}, \tag{A10}$$

which, applying (39), becomes:

$$(u_{11} - \lambda u_{12})(dc^*_t - E_t dc^*_{t+1}) - \frac{u_1 u_{12}}{u_{22} + w u_{12}}(dw_t - E_t dw_{t+1}) = \beta E_t u_1 dr_{t+1}. \tag{A11}$$

Note that (A11) implies, for each $k = 1, 2, \ldots,$

$$E_t dc^*_{t+k} \;=\; dc^*_t - \frac{u_1 u_{12}}{u_{11}u_{22} - u_{12}^2}(dw_t - E_t dw_{t+k}) - \frac{\beta u_1}{u_{11} - \lambda u_{12}}\, E_t \sum_{i=1}^{k} dr_{t+i}. \tag{A12}$$

Combining (2)–(3), differentiating, and evaluating at steady state yields:

$$E_t \sum_{k=0}^{\infty} \frac{1}{(1+r)^k}(dc^*_{t+k} - w dl^*_{t+k} - l dw_{t+k} - dd_{t+k} - a dr_{t+k}) \;=\; (1+r)\, da_t. \tag{A13}$$

Substituting (39) and (A12) into (A13), and solving for dc^*_t, yields:

$$\begin{aligned} dc^*_t \;=\; & \frac{r}{1+r}\,\frac{1}{1+w\lambda}\left[(1+r)da_t + E_t \sum_{k=0}^{\infty}\frac{1}{(1+r)^k}\,(l\, dw_{t+k} + dd_{t+k} + a dr_{t+k})\right] \\ & + \frac{u_1 u_{12}}{u_{11}u_{22} - u_{12}^2}\, dw_t + \frac{1}{1+r}\,\frac{-u_1}{u_{11} - \lambda u_{12}}\, E_t \sum_{k=0}^{\infty} \frac{1}{(1+r)^k}\left[\frac{r\lambda}{1+w\lambda}\, dw_{t+k} - \beta dr_{t+k+1}\right]. \end{aligned} \tag{A14}$$

Proof of Corollary 8

From Definition 3, $\tilde{A}_t \;\equiv\; (1+r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau}(c^*_\tau + w_\tau(\bar{l} - l^*_\tau))$. Evaluated at steady state, $r\tilde{A} = c + w(\bar{l} - l)$, hence (44) follows from (42). Analogously, Definition 2 gives (43).

Note that, from (2)–(3), $\tilde{A}_t = a_t + (1+r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau}(w_\tau \bar{l} + d_\tau)$, hence

$$(1+r)d\tilde{A}_t + \tilde{A}dr_t = (1+r)da_t + a dr_t + E_t \sum_{k=0}^{\infty} \frac{1}{(1+r)^k}\,(\bar{l}\, dw_{t+k} + dd_{t+k}) - \frac{w\bar{l} + d}{r} E_t \sum_{k=1}^{\infty} \frac{1}{(1+r)^k} dr_{t+k}$$

which implies

$$d\hat{A}_t = (1+r)d\tilde{A}_t + \tilde{A}\sum_{k=0}^{\infty}\frac{1}{(1+r)^k}\,dr_{t+k}, \tag{A15}$$

where $d\hat{A}_t$ is as defined in Proposition 7. $d\hat{A}_t$ exceeds $d\tilde{A}_t$ because even holding wealth constant (i.e., $d\tilde{A}_t = 0$), the household can increase consumption in response to a rise in interest rates because the present value of the current consumption path is reduced. Dividing (A15) through by $\tilde{A}$ gives:

$$d\hat{A}_t/\tilde{A} = (1+r)d\log\tilde{A}_t + \sum_{k=0}^{\infty}\frac{1}{(1+r)^k}\,dr_{t+k}, \tag{A16}$$

which can be used in Corollary 8 instead of $d\hat{A}_t/\tilde{A}$.

Proof of Proposition 9

For generalized recursive preferences, the hypothetical one-shot gamble and one-time fee faced by the household are the same as for the case of expected utility. However, the household's optimality conditions for c_t^* and l_t^* (and, implicitly, a_{t+1}^*) are slightly more complicated:

$$u_1(c_t^*, l_t^*) \;=\; \beta\big(E_tV(a_{t+1}^*;\theta_{t+1})^{1-\alpha}\big)^{\alpha/(1-\alpha)}E_tV(a_{t+1}^*;\theta_{t+1})^{-\alpha}V_1(a_{t+1}^*;\theta_{t+1}), \tag{A17}$$

$$u_2(c_t^*, l_t^*) \;=\; -\beta w_t\big(E_tV(a_{t+1}^*;\theta_{t+1})^{1-\alpha}\big)^{\alpha/(1-\alpha)}E_tV(a_{t+1}^*;\theta_{t+1})^{-\alpha}V_1(a_{t+1}^*;\theta_{t+1}). \tag{A18}$$

Note that (A17) and (A18) are related by the usual $u_2(c_t^*, l_t^*) = -w_t u_1(c_t^*; l_t^*)$, and when $\alpha = 0$, (A17) and (A18) reduce to the standard optimality conditions for expected utility.

For an infinitesimal fee $d\mu$ in (7), the change in welfare for the household with generalized recursive preferences is:

$$-V_1(a_t;\theta_t)\,\frac{d\mu}{1+r_t} \;=\; -\beta\big(E_tV(a_{t+1}^*;\theta_{t+1})^{1-\alpha}\big)^{\alpha/(1-\alpha)}E_tV(a_{t+1}^*;\theta_{t+1})^{-\alpha}V_1(a_{t+1}^*;\theta_{t+1})\,d\mu\,, \tag{A19}$$

where the right-hand side of (A19) follows from the envelope theorem.

Turning now to the gamble in (6), the first-order effect of the gamble on household welfare is:

$$\left[u_1\frac{\partial c^*}{\partial\sigma} + u_2\frac{\partial l^*}{\partial\sigma} + \beta\big(E_tV^{1-\alpha}\big)^{\alpha/(1-\alpha)}E_tV^{-\alpha}V_1\cdot\Big(w_t\frac{\partial l^*}{\partial\sigma} - \frac{\partial c^*}{\partial\sigma} + \varepsilon_{t+1}\Big)\right]d\sigma, \tag{A20}$$

where we have dropped the arguments of u_1, u_2, V, and V_1 to simplify notation. As before, optimality of c_t^* and l_t^* implies that the terms involving $\partial c^*/\partial\sigma$ and $\partial l^*/\partial\sigma$ cancel, and $E_tV^{-\alpha}V_1\varepsilon_{t+1} = 0$ because ε_{t+1} is independent of θ_{t+1} and a_{t+1}^*, evaluating the latter at $\sigma = 0$. Thus, the first-order cost of the gamble is zero.

To second order, the effect of the gamble on household welfare is:

$$\begin{aligned}\Bigg\{ & u_{11}\left(\frac{\partial c^*}{\partial\sigma}\right)^2 + 2u_{12}\frac{\partial c^*}{\partial\sigma}\frac{\partial l^*}{\partial\sigma} + u_{22}\left(\frac{\partial l^*}{\partial\sigma}\right)^2 + u_1\frac{\partial^2 c^*}{\partial\sigma^2} + u_2\frac{\partial^2 l^*}{\partial\sigma^2} \\
& + \alpha\beta\big(E_tV^{1-\alpha}\big)^{(2\alpha-1)/(1-\alpha)}\left[E_tV^{-\alpha}V_1\cdot\left(w_t\frac{\partial l^*}{\partial\sigma} - \frac{\partial c^*}{\partial\sigma} + \varepsilon_{t+1}\right)\right]^2 \\
& - \alpha\beta\big(E_tV^{1-\alpha}\big)^{\alpha/(1-\alpha)}E_tV^{-\alpha-1}\left[V_1\cdot\left(w_t\frac{\partial l^*}{\partial\sigma} - \frac{\partial c^*}{\partial\sigma} + \varepsilon_{t+1}\right)\right]^2 \\
& + \beta\big(E_tV^{1-\alpha}\big)^{\alpha/(1-\alpha)}E_tV^{-\alpha}V_{11}\cdot\left(w_t\frac{\partial l^*}{\partial\sigma} - \frac{\partial c^*}{\partial\sigma} + \varepsilon_{t+1}\right)^2 \\
& + \beta\big(E_tV^{1-\alpha}\big)^{\alpha/(1-\alpha)}E_tV^{-\alpha}V_1\cdot\left(w_t\frac{\partial^2 l^*}{\partial\sigma^2} - \frac{\partial^2 c^*}{\partial\sigma^2}\right)\Bigg\}\frac{d\sigma^2}{2}\,.\end{aligned} \tag{A21}$$

The derivatives $\partial c^*/\partial\sigma$ and $\partial l^*/\partial\sigma$ vanish at $\sigma = 0$, the terms involving $\partial^2 c^*/\partial\sigma^2$ and $\partial^2 l^*/\partial\sigma^2$ cancel due to the optimality of c_t^* and l_t^*, and ε_{t+1} is independent of θ_{t+1} and a_{t+1}^* (evaluating the latter at $\sigma = 0$). Thus, (A21) simplifies to:

$$\beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} \left(E_t V^{-\alpha} V_{11} - \alpha E_t V^{-\alpha-1} V_1^2\right) \frac{d\sigma^2}{2}\,. \tag{A22}$$

Equating (A19) to (A22), the Arrow-Pratt coefficient of absolute risk aversion is:

$$\frac{-E_t V^{-\alpha} V_{11} + \alpha E_t V^{-\alpha-1} V_1^2}{E_t V^{-\alpha} V_1}\,. \tag{A23}$$

Since (A23) is already evaluated at $\sigma = 0$, to evaluate it at the nonstochastic steady state, set $a_{t+1} = a$, $\theta_{t+1} = \theta$ to get:

$$\frac{-V_{11}(a;\theta)}{V_1(a;\theta)} + \alpha \frac{V_1(a;\theta)}{V(a;\theta)}\,. \tag{A24}$$

Proof of Proposition 12

Readers working through this proof may find it easier to first consider the case $\alpha = \rho = 0$, that is, expected utility with one-period habits.

The household's first-order conditions for (45) with respect to consumption and labor are given by:

$$u_1 \;=\; \beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V^{-\alpha}\big[V_1 - bV_2\big], \tag{A25}$$

$$u_2 \;=\; -\beta w_t \big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_1, \tag{A26}$$

where we drop the arguments of u and V to reduce notation.

Differentiating (A25) with respect to its first two arguments and applying the envelope theorem yields:

$$V_1 \;=\; \beta(1+r_t)\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_1, \tag{A27}$$

$$V_2 \;=\; -u_1 \;+\; \rho\beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_2. \tag{A28}$$

Equations (A26) and (A27) can be used to solve for V_1 in terms of current-period utility:

$$V_1(a_t, h_t; \theta_t) \;=\; -\frac{(1+r_t)}{w_t}\, u_2(c_t^* - h_t, l_t^*). \tag{A29}$$

To solve for V_{11}, differentiate (A29) with respect to a_t to yield:

$$V_{11}(a_t, h_t; \theta_t) \;=\; -\frac{(1+r_t)}{w_t} \left(u_{12}\, \frac{\partial c_t^*}{\partial a_t} + u_{22}\, \frac{\partial l_t^*}{\partial a_t}\right), \tag{A30}$$

It remains to solve for $\partial c_t^*/\partial a_t$ and $\partial l_t^*/\partial a_t$, which we do in the same manner as in Section 3, except that the dynamics of internal habits require us to solve for $\partial c_\tau^*/\partial a_t$ and $\partial l_\tau^*/\partial a_t$ for all dates $\tau \geq t$ at the same time. To better keep track of these dynamics, we henceforth let a time subscript $\tau \geq t$ denote a generic future date and reserve the subscript t to denote the date of the current period—the period in which the household faces the hypothetical one-shot gamble.

We solve for $\partial l_\tau^*/\partial a_t$ in terms of $\partial c_\tau^*/\partial a_t$ in much the same manner as before, except that the expressions are more complicated due to the persistence of habits and the household's more complicated discounting of future periods. Note first that (A28) can be used to solve for V_2 in terms of current and future marginal utility:

$$V_2(a_t, h_t; \theta_t) = -(1-\rho\beta F)^{-1}\, u_1(c_t^* - h_t^*, l_t^*), \tag{A31}$$

where F denotes the "generalized recursive" forward operator; that is,

$$Fx_\tau \equiv \left(E_\tau V(a_{\tau+1}^*, h_{\tau+1}^*; \theta_{\tau+1}^*)^{1-\alpha}\right)^{\alpha/(1-\alpha)} E_\tau V(a_{\tau+1}^*, h_{\tau+1}^*; \theta_{\tau+1}^*)^{-\alpha} x_{\tau+1}. \tag{A32}$$

The household's intratemporal optimality condition ((A29) combined with (A30)) implies:

$$\begin{aligned} -u_2(c_\tau^* - h_\tau^*, l_\tau^*) \;&=\; w_\tau\big[u_1(c_\tau^* - h_\tau^*, l_\tau^*) \;+\; b\beta E_\tau V_2(a_{\tau+1}^*, h_{\tau+1}^*; \theta_{\tau+1})\big]. && \text{(A33)}\\ &=\; w_\tau\big(1-\beta bF(1-\beta\rho F)^{-1}\big)\, u_1(c_\tau^* - h_\tau^*, l_\tau^*), && \text{(A34)} \end{aligned}$$

Differentiating (A34) with respect to a_t and evaluating at steady state yields:

$$-u_{12}\left(\frac{\partial c_\tau^*}{\partial a_t} - \frac{\partial h_\tau^*}{\partial a_t}\right) - u_{22}\frac{\partial l_\tau^*}{\partial a_t} \;=\; w\left(1-\beta bF(1-\beta\rho F)^{-1}\right)\left[u_{11}\left(\frac{\partial c_\tau^*}{\partial a_t} - \frac{\partial h_\tau^*}{\partial a_t}\right) + u_{12}\frac{\partial l_\tau^*}{\partial a_t}\right], \tag{A35}$$

where we have used the fact that:

$$\frac{\partial}{\partial a_t} Fx_\tau \;=\; F\,\frac{\partial x_\tau}{\partial a_t}\,, \tag{A36}$$

when the derivative is evaluated at steady state. Solving (A35) for $\partial l_\tau^*/\partial a_t$ yields:

$$\begin{aligned} \frac{\partial l_\tau^*}{\partial a_t} \;=\; -\frac{u_{12} + wu_{11} - \beta(\rho u_{12} + (\rho+b)wu_{11})F}{u_{22} + wu_{12}} \times \\ \left[1 - \frac{\beta(\rho u_{22} + (\rho+b)wu_{12})}{u_{22}+wu_{12}}F\right]^{-1}\left(1-bL(1-\rho L)^{-1}\right)\frac{\partial c_\tau^*}{\partial a_t}\,. \end{aligned} \tag{A37}$$

where we've used $h_\tau = bL(1-\rho L)^{-1}c_\tau$ and we assume $\left|\beta(\rho u_{22} + (\rho+b)wu_{12})/(u_{22}+wu_{12})\right| < 1$ to ensure convergence. This solves for $\partial l_t^*/\partial a_t$ in terms of (current and future) $\partial c_\tau^*/\partial a_t$.

We now turn to solving for $\partial c_\tau^*/\partial a_t$. The household's intertemporal optimality (Euler) condition is given by:

$$\frac{1}{w_\tau}\, u_2(c_\tau^* - h_\tau^*, l_\tau^*) \;=\; \beta F\,\frac{1+r_\tau}{w_\tau}\, u_2(c_\tau^* - h_\tau^*, l_\tau^*). \tag{A38}$$

Differentiating (A38) with respect to a_t and evaluating at steady state yields:

$$u_{12}(1-F)\left[1-bL(1-\rho L)^{-1}\right]\frac{\partial c_\tau^*}{\partial a_t} \;=\; -u_{22}(1-F)\frac{\partial l_\tau^*}{\partial a_t}\,. \tag{A39}$$

Using (A37) and noting $FL = 1$ at steady state,[26] (A39) simplifies to:

$$\left[1-\beta(\rho+b)F\right](1-F)\left[1-bL(1-\rho L)^{-1}\right]\frac{\partial c_\tau^*}{\partial a_t} \;=\; 0, \tag{A40}$$

which, from (A39), also implies:

$$\left[1-\beta(\rho+b)F\right](1-F)\,\frac{\partial l_\tau^*}{\partial a_t} \;=\; 0. \tag{A41}$$

[26] To be precise, $FLx_\tau = E_{\tau-1}x_\tau$, but since the household evaluates these expressions from the perspective of the initial period t, $E_tFLx_\tau = E_tx_\tau$. Formally, take the expectation of (39) at time t and then apply $E_tFL = E_t$ to get (40).

Equations (A40) and (A41) hold for all $\tau \geq t$, hence we can invert the $\big[1 - \beta(\rho + b)F\big]$ operator forward to get:

$$(1 - F)\left[1 - bL(1 - \rho L)^{-1}\right] \frac{\partial c_\tau^*}{\partial a_t} \;=\; 0, \tag{A42}$$

$$(1 - F)\, \frac{\partial l_\tau^*}{\partial a_t} \;=\; 0. \tag{A43}$$

Finally, we can apply $(1 - \rho L)$ to both sides of (A42) to get:

$$(1 - F)\left[1 - (\rho + b)L\right] \frac{\partial c_\tau^*}{\partial a_t} \;=\; 0, \tag{A44}$$

which then holds for all $\tau \geq t + 1$. Thus, whatever the initial responses $\partial c_t^*/\partial a_t$ and $\partial l_t^*/\partial a_t$, we must have:

$$E_t \frac{\partial c_{t+1}^*}{\partial a_t} \;=\; (1 + b)\, \frac{\partial c_t^*}{\partial a_t}\,,$$

$$E_t \frac{\partial c_{t+k}^*}{\partial a_t} \;=\; \left(1 + b(\rho + b)^{k-1}\right) \frac{\partial c_t^*}{\partial a_t}\,, \tag{A45}$$

$$\text{and} \quad E_t \frac{\partial l_{t+k}^*}{\partial a_t} \;=\; \frac{\partial l_t^*}{\partial a_t}, \quad k = 1, 2, \ldots \tag{A46}$$

Consumption responds gradually to a surprise change in wealth, while labor moves immediately to its new steady-state level.

From (A45), we can now solve (A39) to get:

$$\frac{\partial l_t^*}{\partial a_t} \;=\; -\lambda\, \frac{\partial c_t^*}{\partial a_t}\,. \tag{A47}$$

where

$$\lambda \;\equiv\; \frac{w(1 - \beta(\rho + b))u_{11} \;+\; (1 - \beta\rho)u_{12}}{(1 - \beta\rho)u_{22} \;+\; w(1 - \beta(\rho + b))u_{12}} \;=\; \frac{u_1 u_{12} - u_2 u_{11}}{u_1 u_{22} - u_2 u_{12}}\,, \tag{A48}$$

where the latter equaltiy follows because $w = -\frac{u_2}{u_1}\frac{1-\beta\rho}{1-\beta(\rho+b)}$ in steady state.

It remains to solve for $\partial c_t^*/\partial a_t$. The household's intertemporal budget constraint implies:

$$E_t \sum_{\tau=t}^{\infty} (1 + r)^{-(\tau - t)}\, \frac{\partial c_\tau^*}{\partial a_t} \;=\; (1 + r) \;+\; w\frac{1 + r}{r}\, \frac{\partial l_t^*}{\partial a_t}\,. \tag{A49}$$

Substituting (A45) and (A47) into (A49) and solving for $\partial c_t^*/\partial a_t$ yields:

$$\frac{\partial c_t^*}{\partial a_t} \;=\; \frac{(1 - \frac{\beta b}{1-\beta\rho})\, r}{1 + (1 - \frac{\beta b}{1-\beta\rho}) w\lambda}\,. \tag{A50}$$

Without habits or labor, an increase in assets would cause consumption to rise by the amount of the income flow from the change in assets—the "golden rule". The presence of habits attenuates this change by the amount $\beta b/(1 - \beta\rho)$ in the numerator, and the consumption response is further attenuated by the household's change in labor income, which is accounted for by the denominator of (A50).

Equations (A29), (A30), (A47), and (A50) allow us to compute the household's coefficient of absolute risk aversion in Proposition 10:[27]

$$\frac{-V_{11}}{V_1} + \alpha\frac{V_1}{V} \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1}\, \frac{(1 - \frac{\beta b}{1-\beta\rho})\, r}{1 + (1 - \frac{\beta b}{1-\beta\rho}) w\lambda} \;+\; \alpha\frac{r\, u_1}{u}\left(1 - \frac{\beta b}{1 - \beta\rho}\right). \tag{A51}$$

[27] In order to express (A51) in terms of u_1 and u_{11} instead of u_2 and u_{22}, we use $V_1 = (1-\beta(\rho+b))u_1/(\beta(1-\beta\rho))$ and differentiate the first-order condition:

$$V_1(a_t, h_t; \theta_t) \;=\; (1 + r_t)\left(1 - \beta b F(1 - \beta\rho F)^{-1}\right) u_1(c_\tau^* - h_\tau, l_\tau^*),$$

with respect to a_t to solve for V_{11}.

The consumption-based coefficient of relative risk aversion is given by:

$$\frac{-AV_{11}}{V_1} + \alpha\frac{AV_1}{V} \;=\; \frac{-u_{11}+\lambda u_{12}}{u_1}\,\frac{(1-\frac{\beta b}{1-\beta\rho})\,c}{1+(1-\frac{\beta b}{1-\beta\rho})w\lambda} \;+\; \alpha\frac{c\,u_1}{u}\left(1-\frac{\beta b}{1-\beta\rho}\right). \tag{A52}$$

Equations (A51) and (A52) have obvious similarities to the corresponding expressions without habits and with expected utility preferences.

Numerical Solution Procedure for Section 7

The equations of the model itself are standard:

$$Y_t = A_t K_{t-1}^{1-\phi} L_t^{\phi}, \tag{A53}$$

$$K_t = (1-\delta)K_{t-1} + Y_t - C_t, \tag{A54}$$

$$C_t^{-\gamma} = \beta E_t (1+r_{t+1}) C_{t+1}^{-\gamma}, \tag{A55}$$

$$\eta L_t^{\chi} / C_t^{-\gamma} = w_t, \tag{A56}$$

$$r_t = (1-\phi)Y_t/K_{t-1} - \delta, \tag{A57}$$

$$w_t = \phi Y_t / L_t, \tag{A58}$$

$$\log A_t = \rho \log A_{t-1} + \varepsilon_t. \tag{A59}$$

In equations (A53)–(A59), note that K_{t-1} denotes the capital stock at the beginning of period t (or the end of period $t-1$), so the notation differs slightly from the main text for compatibility with the numerical algorithm below. To compute risk aversion, we need to append the following auxiliary variables and equations to (A53)–(A59):

$$\lambda_t = (\gamma/\chi) L_t / C_t, \tag{A60}$$

$$C_t^{-\gamma-1}\,\mathrm{DCDA}_t \;=\; \beta E_t (1+r_{t+1}) C_{t+1}^{-\gamma-1}\,\mathrm{DCDA}_{t+1}\left[(1+r_t) - (1+w_t\lambda_t)\,\mathrm{DCDA}_t\right], \tag{A61}$$

$$\mathrm{CARA}_t \;=\; E_t(1+r_{t+1})\gamma C_{t+1}^{-\gamma-1}\,\mathrm{DCDA}_{t+1} \,/\, (C_t^{-\gamma}/\beta), \tag{A62}$$

$$\mathrm{PDVC}_t \;=\; C_t + \beta E_t (C_{t+1}^{-\gamma}/C_t^{-\gamma})\,\mathrm{PDVC}_{t+1}, \tag{A63}$$

$$\mathrm{CRRA}_t \;=\; \mathrm{CARA}_t\,\mathrm{PDVC}_t\,/(1+r_t). \tag{A64}$$

Equation (A60) corresponds to (14), (A61) to (17), (A62) to Proposition 1, and (A63)–(A64) to Definition 2. The variable DCDA_t corresponds to $\partial c_t^*/\partial a_t$. Note that

$$\frac{\partial c_{t+1}^*}{\partial a_t} \;=\; \frac{\partial c_{t+1}^*}{\partial a_{t+1}^*}\left[(1+r_t) - w_t\lambda_t\frac{\partial c_t^*}{\partial a_t} - \frac{\partial c_t^*}{\partial a_t}\right], \tag{A65}$$

which we use in (A61). We use the envelope condition $V_1(a_t;\theta_t) = \beta(1+r_t)E_t V_1(a_{t+1};\theta_{t+1})$ to rewrite $E_t V_1(a_{t+1};\theta_{t+1})$ in (A62), and equations (10)–(11) to rewrite V_1 and V_{11} in terms of derivatives of u.

We solve (A53)–(A64) numerically using the Perturbation AIM algorithm of Swanson, Anderson, and Levin (2006) to compute second- through seventh-order Taylor series approximate solutions to (A53)–(A64) around the nonstochastic steady state. These are guaranteed to be arbitrarily accurate in a neighborhood of the nonstochastic steady state, but importantly also converge globally within the domain of convergence of the Taylor series as the order of the approximation becomes large. Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006) solve a standard real business cycle model like (A53)–(A59) using a variety of numerical methods, including second- and fifth-order perturbation, and find that the perturbation solutions are among the most accurate methods globally, as well as being the fastest to compute. The perturbation solutions we compute for (A53)–(A64) are indistinguishable from one another after the third order over the range of values considered in Figure 2, consistent with Taylor series convergence, so we report only the seventh-order solution in Figure 2.

For the case of Epstein-Zin preferences, we first add equations defining the value function:

$$V_t \;=\; \frac{C_t^{1-\gamma}}{1-\gamma} \;-\; \eta\,\frac{L_t^{1+\chi}}{1+\chi} \;+\; \beta\,\mathrm{VTWIST}_t^{1/(1-\alpha)}, \tag{A66}$$

$$\mathrm{VTWIST}_t \;=\; E_t V_{t+1}^{1-\alpha}. \tag{A67}$$

Next, replace (A55) and (A63) with their Epstein-Zin counterparts:

$$C_t^{-\gamma} \;=\; \beta E_t (1+r_{t+1})\left(V_{t+1}/\mathrm{VTWIST}_t^{1/(1-\alpha)}\right)^{-\alpha} C_{t+1}^{-\gamma}, \tag{A68}$$

$$\mathrm{PDVC}_t \;=\; C_t \;+\; \beta E_t C_{t+1}^{-\gamma}/C_t^{-\gamma}\left(V_{t+1}/\mathrm{VTWIST}_t^{1/(1-\alpha)}\right)^{-\alpha}\mathrm{PDVC}_{t+1}. \tag{A69}$$

Finally, replace (A62) with the corresponding expression from Proposition 9:

$$\mathrm{CARA}_t \;=\; \frac{E_t V_{t+1}^{-\alpha}\left[(1{+}r_{t+1})(\gamma C_{t+1}^{-\gamma-1}\mathrm{DCDA}_{t+1}) \;+\; \alpha(1{+}r_{t+1})^2 C_{t+1}^{-2\gamma}/V_{t+1}\right]}{\mathrm{V1EXP}_t}, \tag{A70}$$

$$\mathrm{V1EXP}_t \;=\; E_t V_{t+1}^{-\alpha}(1{+}r_{t+1})C_{t+1}^{-\gamma}. \tag{A71}$$

The same numerical methods as before can then be applied.

References

Arrow, Kenneth. "Aspects of the Theory of Risk-Bearing," *Yrjö Jahnsson Lectures* (1965), in *Essays in the Theory of Risk Bearing,* ed. Arrow, Kenneth (Chicago: Markum, 1971).

Aruoba, S. Boragan, Jesús Fernández-Villaverde, and Juan Rubio-Ramírez. "Comparing Solution Methods for Dynamic Equilibrium Economies," *Journal of Economic Dynamics and Control* 30 (2006), 2477–508.

Backus, David, Bryan Routledge, and Stanley Zin. "Asset Prices in Business Cycle Analysis," unpublished manuscript (2008), Tepper School of Business, Carnegie Mellon University.

Benveniste, L.M., and J.A. Scheinkman. "On the Differentiability of the Value Function in Dynamic Models of Economics," *Econometrica* 47 (1979), 727–32.

Bernanke, Ben, Mark Gertler, and Simon Gilchrist. "The Financial Accelerator in a Quantitative Business Cycle Framework," *Handbook of Macroeconomics* 1 (1999), 1341–93.

Boldrin, Michele, Lawrence Christiano, and Jonas Fisher. "Habit Persistence and Asset Returns in an Exchange Economy," *Macroeconomic Dynamics* 1 (1997), 312–32.

Boldrin, Michele, Lawrence Christiano, and Jonas Fisher. "Habit Persistence, Asset Returns, and the Business Cycle," *American Economic Review* 91 (2001), 149–66.

Campbell, John and John Cochrane. "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy* 107 (1999), 205–51.

Cochrane, John. *Asset Pricing* (Princeton: Princeton University Press, 2001).

Constantinides, George. "Habit Formation: A Resolution of the Equity Premium Puzzle," *Journal of Political Economy* 98 (1990), 519–43.

Epstein, Larry and Stanley Zin. "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica* 57 (1989), 937–69.

Farmer, Roger. "RINCE Preferences," *Quarterly Journal of Economics* 105 (1990), 43–60.

Flavin, Marjorie and Shinobu Nakagawa. "A Model of Housing in the Presence of Adjustment Costs: A Structural Interpretation of Habit Persistence," *American Economic Review* 98 (2008), 474–95.

Jermann, Urban. "Asset Pricing in Production Economies," *Journal of Monetary Economics* 41 (1998), 257–75.

Kihlstrom, Richard and Leonard Mirman. "Risk Aversion with Many Commodities," *Journal of Economic Theory* 8 (1974), 361–88.

King, Robert, Charles Plosser, and Sergio Rebelo. "Production, Growth, and Business Cycles I: The Basic Neoclassical Model," *Journal of Monetary Economics* 21 (1988), 195–232.

Marinacci, Massimo and Luigi Montrucchio. "Unique Solutions for Stochastic Recursive Utilities," *Journal of Economic Theory* (2010), forthcoming.

Merton, Robert. "An Intertemporal Capital Asset Pricing Model," *Econometrica* 41 (1971), 867–87.

Mortensen, Dale, and Christopher Pissarides. "Job Creation and Job Destruction in the Theory of Unemployment," *Review of Economic Studies* 61 (1994), 397–415.

Pratt, John. "Risk Aversion in the Small and in the Large," *Econometrica* 32 (1964), 122–36.

Rincón-Zapatero, Juan and Carlos Rodríguez-Palmero. “Existence and Uniqueness of Solutions to the Bellman Equation in the Unbounded Case,” *Econometrica* 71 (2003), 1519–55.

Rudebusch, Glenn, and Eric Swanson. “Examining the Bond Premium Puzzle with a DSGE Model,” *Journal of Monetary Economics* 55 (2008), 111–26.

Rudebusch, Glenn, and Eric Swanson. “The Bond Premium in a DSGE Model with Long-Run Real and Nominal Risks,” *Federal Reserve Bank of SF Working Paper* 2008–31 (2009).

Santos, Manuel. “Smoothness of the Policy Function in Discrete Time Economic Models,” *Econometrica* 59 (1991), 1365–82.

Stiglitz, Joseph. “Behavior Towards Risk with Many Commodities,” *Econometrica* 37 (1969), 660–7.

Stokey, Nancy, and Robert Lucas, with Edward Prescott (1989). *Recursive Methods in Economic Dynamics* (Chicago: University of Chicago Press).

Swanson, Eric, Gary Anderson, and Andrew Levin. “Higher-Order Perturbation Solutions to Dynamic, Discrete-Time Rational Expectations Models,” *Federal Reserve Bank of San Francisco Working Paper* 2006–01 (2006).

Tallarini, Thomas. “Risk-Sensitive Business Cycles,” *Journal of Monetary Economics* 45 (2000), 507–32.

Uhlig, Harald. “Leisure, Growth, and Long-Run Risk,” unpublished manuscript University of Chicago (2007).

Van Binsbergen, Jules, Jesus Fernandez-Villaverde, Ralph Koijen, and Juan Rubio-Ramirez. “The Term Structure of Interest Rates in a DSGE Model with Recursive Preferences,” *NBER Working Paper* 15890 (2010).

Weil, Phillipe. “The Equity Premium Puzzle and the Risk-Free Rate Puzzle,” *Journal of Monetary Economics* 24 (1989), 401–21.